Midterm, Analysis III

This is a closed books test. Please solve at least 4 out of 6.

1. Let $f$ be entire, and assume that there exist constants $C > 0$ and $p > 0$ so that $|f(z)| \leq C|z|^p$ holds for all large enough $|z|$. Prove that $f$ is a polynomial.

Solution. Let $R > 0$ be large enough. By Cauchy’s inequalities

$$|f^{(n)}(0)| \leq (n!)CR^{p-n}.$$ 

If $n > p$ then $f^{(n)}(0) = 0$ follows by letting $R \to \infty$.

2. Let $\Omega$ be open, connected and bounded. Assume that a sequence of functions $f_n$, continuous in $\overline{\Omega}$ and holomorphic in $\Omega$, $f_n \in C(\overline{\Omega}) \cap H(\Omega)$, converges uniformly on the boundary $\partial \Omega$. Prove that the sequence converges uniformly in $\Omega$.

Solution. $\sup_{z \in \Omega}|f_n(z) - f_m(z)| \leq \sup_{z \in \partial \Omega}|f_n(z) - f_m(z)|$ by maximum modulus. The sequence is uniformly Cauchy on the boundary.

3. Let $f_n$ be a sequence of one-to-one holomorphic functions in an open connected domain. Assume that $f_n$ converges uniformly on compacts to $f$. Prove that either $f$ is constant or it is one-to-one.

Solution. Suppose $f$ is not constant, and that there exist two points $z_1 \neq z_2$, $z_1, z_2 \in \Omega$, such that $f(z_1) = f(z_2) = w$ Because $f$ is not constant, there exists $r > 0$ and $\delta > 0$ so that $D(z_j, r) \subset \Omega$ and $\inf_{|z_j - z| = r} |f(z) - w| \geq \delta > 0$ hold for $j = 1, 2$. If $f_n$ is close enough to $f$ in $K = \bigcup_{j=1}^2 D(z_j, r)$, then, by Rouché’s thm,

$$\frac{1}{2\pi i} \oint_{|z-z_j|=r} \frac{f'_n(z)}{f_n(z) - w} \, dz = \frac{1}{2\pi i} \oint_{|z-z_j|=r} \frac{f'(z)}{f(z) - w} \, dz,$$

i.e. the number of solutions of the equation $f_n(z) = w$ in $D(z_j, r)$ is the same as the number $n_j$ of solutions of the equation $f(z) = w$ in $D(z_j, r)$. (We take $n$ large enough so that $|f_n - f| < \delta$ is true on both boundaries $|z - z_j| = r$.)
Because of our assumptions about $f$, $n_j \geq 1$, and that contradicts the injectivity of $f_n$, arriving at a contradiction. (The example $f_n = n^{-1}z$ shows that constant functions may arise.)

4. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and assume that $u$ is a weak solution in $H^1_0(\Omega)$ of

$$P(x, Du) = -\sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j u(x)) = f$$

where $a_{ij}(x) = a_{ji}(x)$ are bounded and $P(x, D)$ is uniformly elliptic. Assume that $f \in L^2(\Omega)$.

(a) Prove that $u$ is unique.

(b) Assume, in addition, that $\partial \Omega \in C^2$ and that $a_{ij}$ are continuously differentiable, with bounded first derivatives. Assume that $v \in H^1_0(\Omega)$ is such that there exists a constant $C$ so that

$$\left| \int_{\Omega} v(x) \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j \phi(x)) \, dx \right| \leq C \| \phi \|_{L^2(\Omega)}$$

holds for all $\phi \in C_0^\infty(\Omega)$. Prove that $v \in H^2(\Omega)$.

**Solution.** For (a) we use coercivity and Poincaré’s inequality: If $P(x, Du) = 0$, $u \in H^1_0(\Omega)$ then

$$0 = \int_{\Omega} \sum_{i,j=1}^n a_{ij}(x)(\partial_i u(x))(\partial_j u(x)) \, dx \\
\geq \gamma \int_{\Omega} |\nabla u(x)|^2 \, dx \geq C \int_{\Omega} |u|^2 \, dx.$$

For (b) we have that $v \in H^1_0(\Omega)$, and from the inequality we deduce that $P(x, Du)v \in L^2$. Indeed, the given inequality implies that the map

$$\phi \mapsto \int_{\Omega} v(x)P(x, Du)\phi \, dx$$

is bounded in $L^2(\Omega)$. By the Riesz representation thm, there exists a function $f \in L^2(\Omega)$ so that

$$\int_{\Omega} v(x)P(x, Du)\phi \, dx = \int_{\Omega} f(x)\phi(x) \, dx$$
holds for all $\phi \in C_0^\infty(\Omega)$. The equality

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial_i u(x)\partial_j \phi(x)dx = \int_\Omega f(x)\phi(x)dx$$

follows by integration by parts, which is permitted by a density argument, because $v \in H_0^1(\Omega)$. Then, by definition, it follows that $v \in H_0^1(\Omega)$ is a variational solution of $P(x, D)v = f$, and by the regularity thm, $v \in H^2(\Omega)$.

5. Recall that if $f_j$ is a bounded sequence of functions in $L^1(\mathbb{R}^n)$, i.e. there exists $C > 0$ such that $\|f_j\|_{L^1(\mathbb{R}^n)} \leq C$, and if the sequence is uniformly integrable (uniformly absolutely continuous), $\forall \epsilon > 0$, $\exists \delta > 0$, $|A| \leq \delta \Rightarrow \int_A |f_j|dx \leq \epsilon$, then there exists a subsequence $f_{j_k}$ and a function $f \in L^1(\mathbb{R}^n)$ such that $\lim_{k \to \infty} \int \phi(x)f_{j_k}(x)dx = \int \phi(x)f(x)dx$ holds for any continuous, compactly supported function $\phi$.

(a) Assume that $f \in L^1(\mathbb{R}^n)$. Consider the finite difference quotients

$$(\delta^1_{h}f)(x) = h^{-1}(f(x_1 - h, x_2, \ldots, x_n) - f(x))$$

and assume that there exists a constant $C > 0$, such that

$$\|\delta^1_{h}f\|_{L^1(\mathbb{R}^n)} \leq C$$

for all $|h| \leq 1$. Assume also that the family $\delta^1_{h}f$ is uniformly integrable: $\forall \epsilon > 0$, $\exists \delta > 0$, $|A| \leq \delta \Rightarrow \int_A |\delta^1_{h}f|dx \leq \epsilon$. Show that $\partial_1 f \in L^1(\mathbb{R}^n)$.

(b) Verify that the condition $(1 + |\xi|)\hat{f}(\xi) \in L^1(\mathbb{R}^n)$ is a sufficient condition for the uniform integrability of the family $\delta^1_{h}f$. Here $\hat{f}$ is the Fourier transform.

Solution (a) There exists $g \in L^1(\mathbb{R}^n)$ and a sequence $h_j \to 0$ so that

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} (\delta^1_{h_j}f(x))\phi(x)dx = \int_{\mathbb{R}^n} g(x)\phi(x)dx$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n)$. On the other hand,

$$\int_{\mathbb{R}^n} (\delta^1_{h}f(x))\phi(x)dx = -\int_{\mathbb{R}^n} f(x)(\delta^1_{-h}\phi(x))dx.$$
Now $\delta_{h_j} \phi(x) \to \partial_1 \phi(x)$ pointwise. The sequence is $\delta_{h_j} \phi(x)$ is also uniformly bounded (by $\sup_x |\partial_1 \phi(x)|$), for any fixed $\phi$. Therefore, it follows from the Lebesgue dominated convergence thm that

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} f(x) \delta_{h_j} \phi(x) \, dx = \int_{\mathbb{R}^n} f(x) \partial_1 \phi(x) \, dx.$$ 

Therefore

$$- \int_{\mathbb{R}^n} f(x) \partial_1 \phi(x) \, dx = \int_{\mathbb{R}^n} g(x) \phi(x) \, dx$$

This means that $\partial_1 f = g \in L^1(\mathbb{R}^n)$.

(b)

$$\delta_{h}^1 f(x) = \frac{1}{h} \int_{\mathbb{R}^n} (e^{-ih\xi_1} - 1) e^{ix \cdot \hat{\xi}} \widehat{f}(\xi) \, d\xi.$$ 

Indeed, this follows from the hypothesis and the Fourier inversion formula

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \hat{\xi}} \widehat{f}(\xi) \, d\xi,$$

which holds because $f$ and $\widehat{f}$ are both in $L^1(\mathbb{R}^n)$ by assumption. Then, because

$$\frac{1}{h} |e^{-ih\xi_1} - 1| \leq |\xi_1|,$$

the assumption implies also that

$$|\delta_{h}^1 f(x)| \leq \int_{\mathbb{R}^n} |\xi_1| |\widehat{f}| \, d\xi = C,$$

so the sequence is bounded in $L^\infty(\mathbb{R}^n)$, and hence, a forteriori, it is uniformly integrable.

6.(a) Let $\mu$ be a finite positive Borel measure in $\mathbb{R}^n$, and let

$$P_t[\mu](x) = \int_{\mathbb{R}^n} p_t(x-y) \, d\mu(y),$$

$$p_t(z) = c_n \frac{t}{(t^2 + |z|^2)^{n+1}} e^{-t^2/2}$$

$t > 0$ be its Poisson integral. Let $x_0 \in \mathbb{R}^n$ be such that $D\mu(x_0) = 0$, where $D\mu$ is the Radon-Nikodym derivative with respect to Lebesgue measure. Prove that

$$\lim_{t \to 0} P_t[\mu](x_0) = 0.$$
(b) Give an example of a finite positive Borel measure \( \mu \) in \( \mathbb{R}^n \) such that
\[
D_\mu^+(x) = \limsup_{r \to 0} \frac{1}{|B(x, r)|} \mu(B(x, r))
\]
is finite almost everywhere (with respect to Lebesgue measure) and
\[
P_t[\mu](x) > ND_\mu^+(x)
\]
holds for every \( N \in \mathbb{N} \), almost everywhere with respect to Lebesgue measure.

(a) because \( D_\mu(x_0) = 0 \), for every \( \delta > 0 \), there exists an \( r_0 > 0 \) so that
\[
\mu(B(x_0, r)) \leq \delta |B(x_0, r)|
\]
holds for every \( r \leq r_0 \). We write
\[
\mu = \mu_1 + \mu_2
\]
with \( \mu_1 = \mu_{|B(x_0, r_0)} \). Then
\[
P_t[\mu] = P_t[\mu_1] + P_t[\mu_2].
\]
We claim that
\[
\sup_{t > 0} P_t[\mu_1](x_0) \leq \delta.
\]
Indeed, as we have done in class, if \( \psi(x) = \sum_j c_j \chi_j \) where \( c_j > 0 \) and \( \chi_j \) is the characteristic function of a ball of radius \( r_j \) around \( x_0 \) then
\[
\begin{align*}
t^{-n} \int \psi \left( \frac{x}{t} \right) d\mu_1(x) & \leq \sum_j c_j t^{-n} \mu_1(B(x_0, tr_j)) \\
& \leq \delta \sum_j c_j |B(x_0, r_j)| = \delta \| \psi \|_{L^1(\mathbb{R}^n)}
\end{align*}
\]
holds uniformly in \( t > 0 \). Approximating from below \( p_t(x_0 - x) \) by step functions, radial in \( x_0 - x \), we deduce the claim. On the other hand
\[
P_t[\mu_2](x_0) \leq c_n \frac{t}{(t^2 + r_0^2)^{\frac{n+1}{2}}} \mu(\mathbb{R}^n)
\]
converges to zero as \( t \to 0 \).

(b) We take \( \mu = \delta \). Then
\[
D_\mu^+(x) = 0
\]
for all \( x \neq 0 \) and
\[
P_t[\mu](x) = c_n \frac{t}{(t^2 + x^2)^{\frac{n+1}{2}}}
\]
for \( x \neq 0 \).