Homework 5 – due November 16

Analysis I

1. Let \(1 \leq p < \infty\), \(\frac{1}{p} + \frac{1}{q} = 1\) and let \(\mu\) be a sigma-finite positive measure. Show that the dual of \(L^p(d\mu)\) is \(L^q(d\mu)\). More precisely, show that \(L : L^p(d\mu) \to \mathbb{C}\) is continuous if and only if there exists \(g \in L^q(d\mu)\) such that

\[
L(f) = \int_X fg \, d\mu.
\]

Hint. Start with a finite measure \(\mu\) and set \(\lambda(E) = L(1_E)\). Prove that this is a finite measure, absolutely continuous with respect to \(\mu\) and then use Radon-Nikodym. Or, you may copy from Rudin, but try to think while you are copying.

2. Let \(\mu\) be a measure and define, for any function \(f \in L^p(d\mu)\), the distribution of \(|f|\) to be the function

\[
F(\alpha) = \mu\{x; |f(x)| > \alpha\}.
\]

Prove that, for \(1 \leq p < \infty\),

\[
\|f\|_{L^p(d\mu)}^p = p \int_0^\infty \alpha^{p-1} F(\alpha) d\alpha
\]

and \(\|f\|_{L^\infty(d\mu)} = \inf\{\alpha; F(\alpha) = 0\}\).

Hint: You may use Fubini. Or, you may copy from Rudin, and use the instructions at problem 1.

3. A Radon measure \(\mu\) in \(\mathbb{R}^n\) is said to be doubling if there exists a constant \(D\) such that

\[
\mu(B_5) \leq D \mu(B)
\]

for all balls \(B = B(x, r)\) with \(B_5 = B(x, 5r)\). For any \(f \in L^p(d\mu)\), \(1 \leq p \leq \infty\) define

\[
Mf(x) = \sup_{0 < r \leq 1} \left\{ \frac{\int_{B(x, r)} |f| \, d\mu}{\mu(B(x, r))} \right\}.
\]

(a) If \(p = 1\), prove that there exists a constant \(K\) such that, for all \(\alpha > 0\)

\[
\mu\{x; Mf(x) > \alpha\} \leq \frac{K}{\alpha} \|f\|_{L^1(d\mu)}.
\]
(b) If \( 1 < p \leq \infty \) prove that there exists a positive constant \( C_p \) such that

\[
\|Mf\|_{L^p(d\mu)} \leq C_p\|f\|_{L^p(d\mu)}.
\]

(Hints. For (a) take the set \( A = \{ x; Mf(x) > \alpha \} \) and the family \( F \) of closed balls \( B \) of radii at most one centered at points of \( A \) and such that \( \int_B |f|d\mu > \alpha \mu(B) \). Use the Vitali covering lemma. For (b) consider the function \( f^\alpha(x) = f(x) \) if \( f(x) \geq \frac{\alpha}{2}, f^\alpha(x) = 0 \) otherwise. Note that \( Mf(x) \leq Mf^\alpha(x) + \frac{\alpha}{2} \) and using (a) deduce that the distribution function of \( Mf \) satisfies the inequality

\[
\mu(\{x; Mf > \alpha\}) \leq \frac{2K}{\alpha} \int_{|f| \geq \alpha} |f|d\mu;
\]

then use the previous problem.)

4. Let \( E \subset \mathbb{R}^n \) be Lebesgue measurable and denote \( \lambda \) Lebesgue measure. Prove that

\[
\lim_{r \to 0} \frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))} = 1, \quad \lambda - \text{a.e. } x \in E
\]
and

\[
\lim_{r \to 0} \frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))} = 0, \quad \lambda - \text{a.e. } x \notin E
\]

5. We say that \( f : [a, b] \to \mathbb{R} \) belongs to \( BV([a, b]) \) (bounded variation) if there exists a constant \( C \) such that for any \( a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b \) we have

\[
\sum_{k=1}^{n} |f(t_{k+1}) - f(t_k)| \leq C.
\]

We say that \( f \in AC([a, b]) \) (absolutely continuous) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any finite sequence \( a \leq l_1 < r_1 < l_2 < r_2 \cdots < l_k < r_k \leq b \), if \( \sum_{j=1}^{k} (r_j - l_j) \leq \delta \) then \( \sum_{j=1}^{k} |f(r_j) - f(l_j)| \leq \epsilon \).

(a) Show that \( f \in BV([a, b]) \) if and only if \( f = g - h \) where \( g, h \) are nonincreasing.

(b) Show that if \( f \in BV([a, b]) \) then the usual derivative \( f' \) exists almost everywhere with respect to Lebesgue measure and \( f' \in L^1([a, b]) \).

(c) Show that \( AC([a, b]) \subset BV([a, b]) \).
(d) Show that $f \in AC([a, b])$ if and only if $f \in BV([a, b])$ and

$$f(t) = f(a) + \int_0^t f'(s)ds$$

holds for all $t$.

Hints: Most of this can be found in Rudin. However, the fact that non-decreasing functions have a.e. derivatives, and the derivatives are in $L^1$ is not in Rudin. The hint for (b) is to consider the four numbers

$$
\overline{D}_r f(x) = \limsup_{y \to x, y > x} \frac{f(y) - f(x)}{y - x},
$$

$$
\underline{D}_r f(x) = \liminf_{y \to x, y > x} \frac{f(y) - f(x)}{y - x},
$$

$$
\overline{D} l f(x) = \limsup_{y \to x, y < x} \frac{f(y) - f(x)}{y - x},
$$

$$
\underline{D} l f(x) = \liminf_{y \to 0, y < x} \frac{f(y) - f(x)}{y - x},
$$

and use the Besicovitch covering lemma (in this case of Lebesgue measure it is called the Vitali covering lemma) to prove that for a nondecreasing function the four numbers have to be equal almost everywhere. The covering lemma is: If $F$ is a fine cover of the set $A \subset [a, b]$ formed with closed nodegenerate intervals, then for every open set $U$ containing $A$, $A \subset U$ and every $\epsilon > 0$ there exist finitely many disjoint intervals $I_1, \ldots, I_m$, $I_j \subset U$, $I_j \in F$ so that the outermeasure of $A \setminus \bigcup_{j=1}^m I_j$ is less than $\epsilon$. 

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