Degrees of Homogeneous Models

by

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To get to WEB SITE for these slides:

1. google: "Robert Soare"
2. then click on: /res/vaught/

ABSTRACT:

Degrees of Homogeneous Models

Vaught [1961] defined a model to be homogeneous if every finite partial elementary map can be extended to an automorphism. Goncharov and Peretyatkin found a criterion for a homogeneous model with all types uniformly effectively presented to have a decidable copy. A number of results by researchers at the University of Chicago considerably improve these results in the positive and negative direction. We shall describe some of them.

Vaughtian Models

Vaught [1961]
"Countable Models of Complete Theories"

Soare tutorial

on prime, saturated, homogeneous models.

Differences:

1. We study the *countable* case only.
2. Introduce *homogeneous* models early, get uniqueness of prime, saturated from homog.
3. Study the *tree* of formulas $\mathcal{T}_n(T)$ which generates the n-types $S_n(T) = [\,\mathcal{T}_n(T)\,]$.

Homogeneous Models

Def. $\mathcal{A}$ is *homogeneous* iff for all $\overline{a}$, and $\overline{b}$,

$$(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b}) \implies (\exists G \in \mathrm{Aut}(\mathcal{A}))\, [G(\overline{a}) = \overline{b}].$$

i.e., every finite elementary map $F(\overline{a}) = \overline{b}$ can be extended to an automorphism G of $\mathcal{A}$.

Def. For any model $\mathcal{A} \models T$ define the set of types *realized* in $\mathcal{A}$.

$$\mathbb{T}(\mathcal{A}) = \{p : p \in S(T) \ \ \& \ \ \mathcal{A} \text{ realizes } p\}.$$

Homogeneous Uniqueness Thm. Given a countable complete theory T and homogeneous models $\mathcal{A}$, $\mathcal{B}$ of T with $||\mathcal{A}|| = ||\mathcal{B}||$, then

$$\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}.$$

Spectrum of Homogeneous Models

Algebraically closed fields of characteristic 0

Baldwin-Lachlan sequence of countable models of ACF_0:

$$\overline{Q} \prec \overline{Q}[x_1] \prec \overline{Q}[x_1, x_2] \prec \ldots \quad \overline{Q}[x_i]_{i\in\omega}.$$

prime homogeneous saturated.

Spectrum of Ctble Homogeneous Models

$$\mathcal{A}_0 \qquad \ldots \quad \mathcal{A}_i \quad \ldots \qquad \mathcal{A}_\omega$$

$$S(\mathcal{A}_0) = S^P(T) \quad \subseteq S(\mathcal{A}_i) \quad \subseteq \quad S(\mathcal{A}_\omega) = S(T)$$

prime homogeneous saturated.

but the models are not linearly ordered.

Homogeneous Bounding Degrees

Def. A (Turing) degree $\mathbf{d}$ is *homogeneous bounding* if every complete decidable (CD) theory has a $\mathbf{d}$-decidable homogeneous model.

Def. A degree $\mathbf{d}$ is a *Peano Arithmetic (PA) degree* if $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic.

Thm Csima, Harizanov, Hirschfeldt, Soare
A degree $\mathbf{d}$ is homogeneous bounding iff $\mathbf{d}$ is a PA degree.

Morley's Question

Def. Let $\mathcal{C} \subseteq S(T)$ be a set of computable types of a CD theory T. A $\mathbf{d}$-*basis* for $\mathcal{C}$ is a listing $\{X_n\}_{n\in\omega}$ of $\mathcal{C}$, and a $\mathbf{d}$-computable function $h \leq \mathbf{d}$ such that $\varphi_{h(n)} = X_n$.

Morley's Question. If T is a CD theory and $\mathcal{A}$ is a homogeneous model of T with a $\mathbf{0}$-basis X for $\mathbb{T}(\mathcal{A})$ does $\mathcal{A}$ have a decidable copy $\mathcal{B}$?

Note. True for prime and saturated models.

Twin Matrix Picture:

Given model $\mathcal{A}$ with types $\mathbb{T}(\mathcal{A}) = \{A_i\}_{i\in\omega}$

A_0

A_1

A_2

$\vdots$

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Construct model $\mathcal{B}$ with $\mathbb{T}(\mathcal{B}) = \mathbb{T}(\mathcal{A})$.

B_0

B_1

B_2

$\vdots$

EEF and Decidable Copies

Let $\mathcal{A}$ be a homogeneous model of a CD theory T and type spectrum m $\mathbb{T}(\mathcal{A})$ has a **0**-basis $X = \{p_i\}_{i \in \omega}$.

f is an *effective extension function (EEF)* for X if

- for every n-type $p_i(\overline{x}) \in X \cap S_n(T)$
- $p_i(\overline{x}) \subseteq p_{f(i)}(\overline{x}, x_n) \in X \cap S_{n+1}(T)$.

Positive Thm. [Goncharov, Peretyatkin]. Let T be a CD theory and $\mathcal{A} \models T$ homogeneous. TFAE:

(i) $\mathcal{A}$ has a decidable copy $\mathcal{B}$.

(ii) Some 0-basis for $\mathbb{T}(\mathcal{A})$ has EEF.

Degrees of Homogeneous Models

Thm 1. [Karen Lange]
[**Homogeneous Low Basis Thm**]. Given:

- a CD theory T;
- a homogeneous model $\mathcal{A} \models T$;
- a $\mathbf{0}'$-basis $X = \mathbb{T}(\mathcal{A})$.

then there is a copy $\mathcal{B} \cong \mathcal{A}$ which is *low*.
(i.e., $D^e(\mathcal{B})' \equiv_{\mathrm{T}} 0'$.)

Coroll. [Prime Low Basis Thm, Csima]
Every complete atomic decidable (CAD) theory T has a a low prime model $\mathcal{A}$.

Prf. If T is CAD, then *any* prime model $\mathcal{A} \models T$ has a $0'$-basis $X = \mathbb{T}(A) = \mathbb{T}^P(T)$.

Nonlow$_2$ Bounding

Thm 2. [**Karen Lange**].
[**Homogeneous Bounding Theorem**] Given:

- A CD theory T;
- A homogeneous model $\mathcal{A} \models T$;
- A $\mathbf{0}$-basis $X = S(\mathcal{A})$;
- A degree $\mathbf{d} \leq \mathbf{0}'$ which is *nonlow*$_2$ ($\mathbf{d}'' > \mathbf{0}''$).

Then there is a $\mathbf{d}$-decidable copy $\mathcal{B} \cong \mathcal{A}$.

Note. Using Lange Homogeneous Low Basis Thm 1, strengthen to the $\mathbf{0}'$-uniform case.

Cor. [**Csima, Hirschfeldt, Knight, Soare**] If $\mathbf{d} \leq \mathbf{0}'$ is *nonlow*$_2$ then $\mathbf{d}$ is prime bounding.

Domination and Escape

Def. A fn h *dominates* a fn f if

$$(\forall^\infty x)[\ f(x) < h(x)\ ],$$

and otherwise f *escapes* h,

$$(\exists^\infty x)[\ h(x) \leq f(x)\ ].$$

Escape Property

$$D \leq_{\mathrm{T}} \emptyset' \ \textit{nonlow}_2 \iff$$
$$(\forall h \leq \mathbf{0}')(\exists f \leq_{\mathrm{T}} D)(\exists^\infty t)[\ h(t) \leq f(t)\ ]$$

$S(T)$ Types All Computable (TAC)

Thm 3. [**Karen Lange**].
[**Homogeneous Full Basis Theorem**]
Let T be a CD theory with types all computable (TAC). Let homogeneous $\mathcal{A} \models T$ have a $\mathbf{0}$-basis. Then

$$\{\mathbf{d} : \mathbf{0} < \mathbf{d}\} \subseteq \{ \deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A} \}.$$

Note. Like the Csima-Hirschfeldt Full Basis Thm for prime models of a CAD theory T with TAC. Neither theorem implies the other.

Saturated Models

Def. Let T be a countable complete theory, $\mathcal{A} \models T$ countable.

(i) $\mathcal{A}$ is *saturated* if every 1-type $p(\overline{a}, x)$ over a finite set of elements $\overline{a} \in A$ is realized in $\mathcal{A}$.

(ii) If $\mathcal{A}$ is homogeneous then $\mathcal{A}$ is saturated iff $\mathbb{T}(\mathcal{A}) = S(T)$ (*i.e.*, $\mathcal{A}$ is *weakly saturated*).

Def. A degree $\mathbf{d}$ is *saturated bounding* if for every CD+TAC theory T there is a saturated model $\mathcal{A}$ of T which is $\mathbf{d}$-decidable.

Thm. Every degree $\mathbf{d}$ which is *high* ($\mathbf{d}' \geq \mathbf{0}''$) is saturated bounding.

Thm(Millar). Degree $\mathbf{0}$ is not saturated bounding.

Uniform Escape Property

Thm (Ken Harris). There is a CD + TAC theory T with no *low* saturated model.

Def. (Harris)*** A degree $\mathbf{d}$ has the *Uniform Escape Property* if there is an $h \leq_{\mathrm{T}} \mathbf{0}$ such that

$$(\forall e)[\Phi_e^{\mathbf{d}} \text{ total } \implies (\exists^\infty x)[\ \Phi_e^{\mathbf{d}}(x) \ \leq \ \Phi_{h(e)}(x)\ ]\ ].$$

Thm (Ken Harris). For *c.e.* degrees $\mathbf{d}$, TFAE:

(a) $\mathbf{d}$ is *low* ($\mathbf{d}' = \mathbf{0}'$).

(b) $\mathbf{d}$ has the Uniform Escape Property.

Extending Negative Results

Def. A degree **d** is *low*$_n$ if $\mathbf{d}^{(n)} = 0^{(n)}$.

Thm (K. Harris) For $n \geq 1$ TFAE:

(i) A is low$_n$.

(ii) A has n-UEP.

Def. A refinement of n-UEP is the *aligned escape property (AEP).*

Thm. All low$_n$ c.e. degrees have AEP.

Thm. A has n-AEP $\Longrightarrow$ A not saturated bounding.

Coroll. No low$_n$ c.e. degree is saturated bounding.

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$\mathbf{Low}_n$ Degrees and Escape Functions

There is a hierarchy of properties characterized by less effective procedures, **Uniform Escape Property** n**-UEP**, starting with (1-UEP)=(UEP), such that

Thm. For all degrees $\mathbf{d}$ and all $n \geq 1$ TFAE:

(i) $\mathbf{d}$ is $\mathbf{L}_n$ $(\mathbf{d}^{(n)} = \mathbf{0}^{(n)})$.

(ii) $\mathbf{d}$ has (n-UEP).

n-Uniform Escape Property

Def. Degree $\mathbf{d}$ has the *n-Uniform Escape Property* (n-UEP) if for any set $A \in \mathbf{d}$:

There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e[\, h_{e,\overline{y}}\,],\ \overline{y} \in \omega\,]$

such that for any u.e. family $\{\Phi^A_{e,\overline{y}}\}_{\overline{y}\in\omega}$ with

$$(Q_1)(Q_2)\ldots\ [\Phi^A_{e,\overline{y}} \text{ total }\ ] \quad \Longrightarrow$$
$$(Q_1)(Q_2)\ldots\ [\ h_{e,\overline{y}} \text{ total } \ \& \ \ \text{escapes } \Phi^A_{e,\overline{y}}\ ]$$

where $Q_1, Q_2, \ldots$ are certain quantifiers over the $y's \in \overline{y}$ matched in hypothesis and conclusion.

Noncomputability and Lowness

Gödel [1931] Incompleteness

Turing [1936] Incomputability and undecidability *Entscheidungsproblem*.

[2006] Analyze these undecidable (noncomputable) sets especially the simplest, *i.e.*, sets of low information content.

Willian Rainey Harper Dissertation Award. Never won by a math grad student.

[May, 2006] Ken Harris won Harper Award.