Degrees of Homogeneous Models

by

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To get to WEB SITE for these slides:

1. google: “Robert Soare”
2. then click on: /res/vaught/
ABSTRACT:

Degrees of Homogeneous Models

Vaught [1961] defined a model to be homogeneous if every finite partial elementary map can be extended to an automorphism. Goncharov and Peretyatkin found a criterion for a homogeneous model with all types uniformly effectively presented to have a decidable copy. A number of results by researchers at the University of Chicago considerably improve these results in the positive and negative direction. We shall describe some of them.
Vaughtian Models

Vaught [1961]
“Countable Models of Complete Theories”

Soare tutorial

on prime, saturated, homogeneous models.

Differences:
1. We study the countable case only.
2. Introduce homogeneous models early, get uniqueness of prime, saturated from homog.
3. Study the tree of formulas $T_n(T)$ which generates the $n$-types $S_n(T) = [T_n(T)]$. 
**Homogeneous Models**

**Def.** $A$ is *homogeneous* iff for all $\bar{a}$, and $\bar{b}$,

$$(A, \bar{a}) \equiv (A, \bar{b}) \implies (\exists G \in \text{Aut}(A)) [G(\bar{a}) = \bar{b}].$$

i.e., every finite elementary map $F(\bar{a}) = \bar{b}$ can be extended to an automorphism $G$ of $A$.

**Def.** For any model $A \models T$ define the set of types realized in $A$.

$$T(A) = \{p : p \in S(T) \; \& \; A \text{ realizes } p\}.$$  

**Homogeneous Uniqueness Thm.** Given a countable complete theory $T$ and homogeneous models $A, B$ of $T$ with $||A|| = ||B||$, then

$$T(A) = T(B) \implies A \cong B.$$
Spectrum of Homogeneous Models

(Algebraically closed fields of characteristic 0)

Baldwin-Lachlan sequence of countable models of $\text{ACF}_0$:

$\mathbb{Q} \prec \mathbb{Q}[x_1] \prec \mathbb{Q}[x_1, x_2] \prec \cdots \prec \mathbb{Q}[x_i]_{i \in \omega}$.

prime homogeneous saturated.

Spectrum of Ctable Homogeneous Models

$\mathcal{A}_0 \quad \ldots \quad \mathcal{A}_i \quad \ldots \quad \mathcal{A}_\omega$

$S(\mathcal{A}_0) = S^p(T) \subseteq S(\mathcal{A}_i) \subseteq S(\mathcal{A}_\omega) = S(T)$

prime homogeneous saturated.

but the models are not linearly ordered.
Homogeneous Bounding Degrees

**Def.** A (Turing) degree $d$ is *homogeneous bounding* if every complete decidable (CD) theory has a $d$-decidable homogeneous model.

**Def.** A degree $d$ is a *Peano Arithmetic (PA) degree* if $d$ is the degree of a complete extension of Peano Arithmetic.

**Thm Csima, Harizanov, Hirschfeldt, Soare**
A degree $d$ is homogeneous bounding iff $d$ is a PA degree.
Morley’s Question

**Def.** Let $\mathcal{C} \subseteq S(T)$ be a set of computable types of a CD theory $T$. A $d$-basis for $\mathcal{C}$ is a listing $\{X_n\}_{n \in \omega}$ of $\mathcal{C}$, and a $d$-computable function $\varphi \leq d$ such that $\varphi h(n) = X_n$.

**Morley’s Question.** If $T$ is a CD theory and $\mathcal{A}$ is a homogeneous model of $T$ with a 0-basis $X$ for $T(\mathcal{A})$ does $\mathcal{A}$ have a decidable copy $\mathcal{B}$?

**Note.** True for prime and saturated models.
Twin Matrix Picture:

Given model $\mathcal{A}$ with types $\mathcal{T}(\mathcal{A}) = \{A_i\}_{i \in \omega}$

$A_0$
$A_1$
$A_2$
$\vdots$

Construct model $\mathcal{B}$ with $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{A})$.

$B_0$
$B_1$
$B_2$
$\vdots$
Let \( \mathcal{A} \) be a homogeneous model of a CD theory \( T \) and type spectrum \( \mathbb{T}(\mathcal{A}) \) has a \( 0 \)-basis \( X = \{ p_i \}_{i \in \omega} \).

\( f \) is an effective extension function (EEF) for \( X \) if

- for every \( n \)-type \( p_i(\overline{x}) \in X \cap S_n(T) \)
- \( p_i(\overline{x}) \subseteq p_{f(i)}(\overline{x}, x_n) \in X \cap S_{n+1}(T) \).

**Positive Thm. [Goncharov, Peretyatkin].**
Let \( T \) be a CD theory and \( \mathcal{A} \models T \) homogeneous.

TFAE:

(i) \( \mathcal{A} \) has a decidable copy \( \mathcal{B} \).

(ii) Some \( 0 \)-basis for \( \mathbb{T}(\mathcal{A}) \) has EEF.
Degrees of Homogeneous Models

**Thm 1. [Karen Lange]**

[Homogeneous Low Basis Thm]. Given:

- a CD theory $T$;
- a homogeneous model $\mathcal{A} \models T$;
- a $0'$-basis $X = T(\mathcal{A})$.

then there is a copy $\mathcal{B} \cong \mathcal{A}$ which is low.
(i.e., $D^p(\mathcal{B})' \equiv_T 0'$.)

**Coroll. [Prime Low Basis Thm, Csima]**

Every complete atomic decidable (CAD) theory $T$ has a a low prime model $\mathcal{A}$.

**Prf.** If $T$ is CAD, then any prime model $\mathcal{A} \models T$ has a $0'$-basis $X = T(\mathcal{A}) = T^p(T)$. 
Nonlow\textsubscript{2} Bounding

Thm 2. [Karen Lange].
[Homogeneous Bounding Theorem] Given:

- A CD theory $T$;
- A homogeneous model $\mathcal{A} \models T$;
- A 0-basis $X = S(\mathcal{A})$;
- A degree $d \leq 0'$ which is nonlow\textsubscript{2} ($d'' > 0''$).

Then there is a $d$-decidable copy $\mathcal{B} \cong \mathcal{A}$.

Note. Using Lange Homogeneous Low Basis Thm 1, strengthen to the 0'-uniform case.

Cor. [Csima, Hirschfeldt, Knight, Soare] If $d \leq 0'$ is nonlow\textsubscript{2} then $d$ is prime bounding.
Domination and Escape

**Def.** A fn $h$ dominates a fn $f$ if

$$(\forall^\infty x)[ f(x) < h(x) ],$$

and otherwise $f$ escapes $h$,

$$(\exists^\infty x)[ h(x) \leq f(x) ].$$

**Escape Property**

$$D \leq_T \emptyset' \text{ nonlow}_{2} \iff (\forall h \leq 0')(\exists f \leq_T D)(\exists^\infty t)[ h(t) \leq f(t) ]$$
Thm 3. [Karen Lange].

[Homogeneous Full Basis Theorem]
Let $T$ be a CD theory with types all computable (TAC). Let homogeneous $\mathcal{A} \models T$ have a $0$-basis. Then

$$\{ d : 0 < d \} \subseteq \{ \deg(B) : B \cong \mathcal{A} \}.$$ 

Note. Like the Csima-Hirschfeldt Full Basis Thm for prime models of a CAD theory $T$ with TAC. Neither theorem implies the other.
**Saturated Models**

**Def.** Let $T$ be a countable complete theory, $\mathcal{A} \models T$ countable.

(i) $\mathcal{A}$ is *saturated* if every 1-type $p(\bar{a}, x)$ over a finite set of elements $\bar{a} \in A$ is realized in $\mathcal{A}$.

(ii) If $\mathcal{A}$ is homogeneous then $\mathcal{A}$ is saturated iff $T(\mathcal{A}) = S(T)$ (i.e., $\mathcal{A}$ is *weakly saturated*).

**Def.** A degree $d$ is *saturated bounding* if for every CD+TAC theory $T$ there is a saturated model $\mathcal{A}$ of $T$ which is $d$-decidable.

**Thm.** Every degree $d$ which is *high* ($d' \geq 0''$) is saturated bounding.

**Thm (Millar).** Degree $0$ is not saturated bounding.
Uniform Escape Property

Thm (Ken Harris). There is a CD + TAC theory $T$ with no low saturated model.

Def. (Harris) A degree $d$ has the Uniform Escape Property if there is an $h \leq_T 0$ such that

$$(\forall e)[\Phi^d_e \text{ total } \implies (\exists x)[\Phi^d_e(x) \leq \Phi^d_{h(e)}(x)]]$$.

Thm (Ken Harris). For c.e. degrees $d$, TFAE:

(a) $d$ is low ($d' = 0'$).

(b) $d$ has the Uniform Escape Property.
Extending Negative Results

Def. A degree $d$ is $\text{low}_n$ if $d^{(n)} = 0^{(n)}$.

Thm (K. Harris) For $n \geq 1$ TFAE:

(i) $A$ is $\text{low}_n$.

(ii) $A$ has $n$-UEP.

Def. A refinement of $n$-UEP is the aligned escape property (AEP).

Thm. All $\text{low}_n$ c.e. degrees have AEP.

Thm. $A$ has $n$-AEP $\implies A$ not saturated bounding.

Coroll. No $\text{low}_n$ c.e. degree is saturated bounding.
Def. A degree $d$ is low$_n$ if $d^{(n)} = 0^{(n)}$.

Thm (K. Harris) For $n \geq 1$ TFAE:

(i) $A$ is low$_n$.

(ii) $A$ has $n$-UEP.

Def. A refinement of $n$-UEP is the aligned escape property (AEP).

Thm. All low$_n$ c.e. degrees have AEP.

Thm. $A$ has $n$-AEP $\implies A$ not saturated bounding.

Coroll. No low$_n$ c.e. degree is saturated bounding.
There is a hierarchy of properties characterized by less effective procedures, Uniform Escape Property \( n\text{-UEP} \), starting with (1-UEP)=(UEP), such that

**Thm.** For all degrees \( d \) and all \( n \geq 1 \) TFAE:

(i) \( d \) is \( L_n \) (\( d^{(n)} = 0^{(n)} \)).

(ii) \( d \) has \( (n\text{-UEP}) \).
**n-Uniform Escape Property**

**Def.** Degree $d$ has the *n-Uniform Escape Property* ($n$-UEP) if for any set $A \in d$:

There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e [h_e, \eta], \eta \in \omega$ such that for any u.e. family $\{\Phi^A_{e, \eta}\}_{\eta \in \omega}$ with

$$(Q_1)(Q_2) \ldots [\Phi^A_{e, \eta} \text{ total }] \implies (Q_1)(Q_2) \ldots [h_e, \eta \text{ total } \& \text{ escapes } \Phi^A_{e, \eta}]$$

where $Q_1, Q_2, \ldots$ are certain quantifiers over the $\eta$'s $\in \eta$ matched in hypothesis and conclusion.
Noncomputability and Lowness

Gödel [1931] Incompleteness

Turing [1936] Incomputability and undecidability Entscheidungsproblem.

[2006] Analyze these undecidable (noncomputable) sets especially the simplest, \textit{i.e.}, sets of low information content.

Willian Rainey Harper Dissertation Award.
Never won by a math grad student.