Computability of Vaughtian Models:

Lecture 4: Degrees of Homogeneous Models

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Homogeneous Models

**Def.** \( \mathcal{A} \) is homogeneous iff for all \( \bar{a} \), and \( \bar{b} \),
\[
(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b}) \implies (\exists G \in \text{Aut}(\mathcal{A}))[G(\bar{a}) = \bar{b}].
\]
i.e., every finite elementary map \( F(\bar{a}) = \bar{b} \) can be extended to an automorphism \( G \) of \( \mathcal{A} \).

**Def.** For any model \( \mathcal{A} \models T \) define the set of types realized in \( \mathcal{A} \).
\[
T(\mathcal{A}) = \{ p : p \in S(T) \ & \mathcal{A} \text{ realizes } p \}.
\]

**Homogeneous Uniqueness Thm.** Given a countable complete theory \( T \) and homogeneous models \( \mathcal{A}, \mathcal{B} \) of \( T \) with \( ||\mathcal{A}|| = ||\mathcal{B}|| \), then
\[
T(\mathcal{A}) = T(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}.
\]
Proof of Homogeneous Uniqueness

**Proof.** Fix $A$ and $B$ be homogeneous, countable, with $\mathcal{T}(A) = \mathcal{T}(B)$. Suffices to define $\omega$-back and forth $F$ between $A$ and $B$. $T$ is complete so $A \equiv B$. Add the empty map $\emptyset$ to $F$.

Given any elementary map $f \in F$, $f(\pi) = \bar{b}$ and any $c \in A$ let $p$ be the $(n+1)$-type of $(\pi, c)$ in $A$. There is some $(n+1)$-tuple $(\bar{b}', d')$ satisfying $p$ in $B$ because $\mathcal{T}(A) = \mathcal{T}(B)$.

Hence, $(B, \bar{b}) \equiv (B, \bar{b}')$. By homogeneity of $B$ there is some $d \in B$ such that

$$(A, \pi, c) \equiv (B, \bar{b}', d') \equiv (B, \bar{b}, d).$$

Extend $f$ to $g = f \cup \{(c, d)\}$ and add $g$ to $F$. 

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Spectrum of Homogeneous Models

Algebraically closed fields of characteristic 0

Baldwin-Lachlan sequence of countable models of $\text{ACF}_0$:

$\mathbb{Q} \prec \mathbb{Q}[x_1] \prec \mathbb{Q}[x_1, x_2] \prec \ldots \prec \mathbb{Q}[x_i]_{i \in \omega}$

prime homogeneous saturated.

Spectrum of Ctbble Homogeneous Models

$\mathcal{A}_0 \ldots \mathcal{A}_i \ldots \mathcal{A}_\omega$

$S(\mathcal{A}_0) = S^p(T) \subseteq S(\mathcal{A}_i) \subseteq S(\mathcal{A}_\omega) = S(T)$

prime homogeneous saturated.

but the models are not linearly ordered.
Homogeneous Bounding Degrees

**Def.** A degree $d$ is *homogeneous bounding* if every CD theory has a $d$-decidable homogeneous model.

**Def.** A degree $d$ is a *Peano Arithmetic (PA) degree* if $d$ is the degree of a complete extension of Peano Arithmetic.

**Thm Csima, Harizanov, Hirschfeldt, Soare.** A degree $d$ is homogeneous bounding iff $d$ is a PA degree.
**PA Computes Homogeneous Model**

**Thm.** Any countable theory $T$ has a homogeneous model $\mathcal{A}$.

Build $\mathcal{A} = \bigcup \mathcal{A}_n$ and elementary chain. At level $n$ list all finite partial elementary maps over $\mathcal{A}_n$ and in $\mathcal{A}_{n+1}$ add new constants to guarantee one apoint extension of each map.

This repeatedly uses Lindenbaum’s Lemma (that every consistent set of sentences can be extended to a complete theory) which is equivalent to finding paths through trees.

Lindenbaum’s Lemma can be carried out effectively in a degree $d$ iff $d$ is a PA degree.
Homogeneous Bounding is PA

Let $U$ (or $V$) be the set of Gödel numbers of sentences provable (or refutable) from PA. Any separating set for $U$ and $V$ has a PA degree.

**Key Idea.** Build theory $T$ s.t. if $A \models T$ is homogeneous, then the atomic diagram of $D(A)$ can compute a separating set for $U$ and $V$.

$L(T)$ has infinitely many unary predicate symbols $\{P_i\}_{i \in \omega}$, infinitely many binary predicate symbols $\{R_i\}_{i \in \omega}$, a unary predicate symbol $D$, and a binary predicate symbol $E$.
Morley’s Four Properties

Morley [1976, p. 236] noted:

\( P1. \) There is a decidable model \( A. \)

\( P2. \) There is a computable listing of \( T(A). \)

\( P3. \) \( T(A) \) satisfies TAC (types all computable).

\( P4. \) The theory \( T \) is CD.

Morley noted the obvious:

\[ P1 \implies P2 \implies P3 \implies P4. \]

\[ P4 \not\implies P3. \]

\[ P3 \not\implies P2. \]
**Morley’s Question**

**Def.** Let $\mathcal{C} \subseteq S(T)$ be a set of types of a CD theory $T$. If there exists some uniformly computable listing $X = \{p_j\}_{j \in \omega}$ of $\mathcal{C}$ we call $X$ a $0$-basis for $\mathcal{C}$.

**Morley’s Question.** If $T$ is a CD theory and $\mathcal{A}$ is a homogeneous model of $T$ with a $0$-basis $X$ for $\mathcal{T}(\mathcal{B})$ does $\mathcal{A}$ have a decidable copy $\mathcal{B}$?

By the Homogeneous Uniqueness Thm this is equivalent to finding a decidable homogeneous model $\mathcal{B}$ of $T$ with $\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{B})$.

**Note.** True for prime and saturated models.
Let \( \mathcal{A} \) be a homogeneous model of a CD theory \( T \) whose type spectrum \( \mathcal{T}(\mathcal{A}) \) has a \( 0 \)-basis \( X = \{ p_i \}_{i \in \omega} \).

(i) A function \( f \) is an extension function (EF) for \( X \) if for every \( n \),
- for every \( n \)-type \( p_i(\bar{x}) \in X \cap S_n(T) \)
- and every \((n + 1)\)-ary \( \theta_j(\bar{x}, x_n) \in F_{n+1}(T) \) consistent with \( p_i(\bar{x}) \)
  \[ p_i(\bar{x}) \cup \{ \theta_j(\bar{x}, x_n) \} \subseteq p_{f(i,j)}(\bar{x}, x_n). \]

(ii) If \( f \) is also computable then \( f \) is an effective extension function (EEF).
Picture slide:

pix: Have one matrix, w/ p1 as 1-type on first row, and 2-ary fmls $\theta$. Now move marker along 2-rows until it settles on right answer.
Monotone Extension Function

**Def.** [Monotone Function on $X$] A function $f(i)$ on $X$ is a monotone function on $X$ if there is a computable function $\hat{f}(i,s)$ such that,

(i) $f(i) = \lim_{s} \hat{f}(i,s)$, and

(ii) $p_{\hat{f}(i,s)}\mid s \subseteq p_{\hat{f}(i,s+1)}\mid (s + 1)$.

In this case we write $f(i) = mlim_{s} \hat{f}(i,s)$.

**IDEA.** We build a computable type $q = \cup_{s} p_{\hat{f}(i,j,s)}\mid s$ as the union of a monotone sequence $\{p_{\hat{f}(i,j,s)}\mid s\}$.

**Thm.** If $X = \{p_{i}\}_{i \in \omega}$ and $Y = \{q_{i}\}_{i \in \omega}$ are 0-bases for $T(A)$ there is a monotonic function $g$ on $X$ such that $p_{g(i)} = q_{i}$. 
EEF and Decidable Copies

Positive Thm. [Goncharov, Peretyatkin]. Let \( T \) be a CD theory and \( \mathcal{A} \models T \) homogeneous. TFAE:

(i) \( \mathcal{A} \) has a decidable copy \( \mathcal{B} \).

(ii) Every 0-basis \( X = S(\mathcal{A}) \) has MEF.

(iii) Some 0-basis for \( S(\mathcal{A}) \) has MEF.

(iv) Some 0-basis for \( S(\mathcal{A}) \) has EEF.

Proof. (i) \( \Rightarrow \) (iv), obvious.

(ii) \( \iff \) (iii) \( \iff \) (iv), easy.

(iv) \( \Rightarrow \) (i), main import of the theorem.
**Corollaries: Prime and Saturated**

**Application Thm.** Let $T$ be CD and $A \models T$ have 0-basis $X = S(A)$. If $A$ is either:

- prime, or
- saturated,

then $X$ has MEF. (Hence, $\exists$ decidable $B \cong A$.)

**Coroll [Harrington, Goncharov-Nurtazin].** If $T$ is a complete atomic decidable (CAD) theory and $S^p(T)$ has a 0-basis then $T$ has a decidable prime model.

**Coroll [Morley, Millar].** If $T$ is a complete decidable (CD) theory and $S(T)$ has a 0-basis, then $T$ has a decidable saturated model.
PIX: Ease to see prime case by movable marker.
Satur. easy also.
Picture of Prime Model and MEF
Negative Thm.
[Goncharov, Peretyatkin, Millar].
There exists:

- a CD theory $T$,
- a homogeneous model $\mathcal{A} \models T$,
- a 0-basis $X = S(\mathcal{A})$,

with no decidable copy $\mathcal{B} \cong \mathcal{A}$.

Proof.
Construct 0-basis $X = S(\mathcal{A})$ with no MEF.
Pix. Push opponent’s MEF fn off to infty.
Homogeneity Conditions

Existence Thm for Ctable Homogeneous Models [Goncharov, Peretyat’kin].
Given CD theory $T$ and ctable $S \subseteq S(T)$.

$$(\exists \text{ homogeneous } A \models T) \ [ S(A) = S ]$$

\[ \iff \]

1. $S$ is closed under taking subtypes, and
2. $S$ is closed under permutations of variables,
3. (EP) If $p(x_1, ..., x_n) \in S$ and $\theta(x_1, ..., x_{n+1})$ are consistent, then there exists an $(n + 1)$-type $q \in S$ such that $p \cup \{ \theta \} \subseteq q$, and
4. (TAP) For any two types $p_1(\bar{x}, y)$, $p_2(\bar{x}, z) \in S$ such that $p_1 \downarrow \bar{x} = p_2 \downarrow \bar{x}$, there exists a type $q(\bar{x}, y, z)$ containing $p_1$ and $p_2$. 
Degrees of Homogeneous Models

Thm 1. [Karen Lange]  
[Homogeneous Low Basis Thm]. Given:

- A CD theory $T$;
- A homogeneous model $A \models T$;
- A $0'$-basis $X = S(A)$.

Then there is a copy $B \cong A$ which is low.  
(Namely, $D^r(B)' \equiv_T 0'$.)

Coroll.  [Prime Low Basis Thm, Csima]  
Every complete atomic decidable (CAD) theory $T$ has a a low prime model $A$.

Prf. If $T$ is CAD, then any prime model $A \models T$ has a $0'$-basis $X = S(A) = S^P(T)$. 
Thm 2. [Karen Lange].

[Homogeneous Full Basis Theorem]
Let $T$ be a CD theory with types all computable (TAC). Let homogeneous $\mathcal{A} \models T$ have a $\mathbf{0}$-basis. Then

$$\{ d : 0 < d \} \subseteq \{ \deg(B) : B \cong \mathcal{A} \}.$$ 

Note. Like the Csima-Hirschfeldt Full Basis Thm for prime models of a CAD theory $T$ with TAC. Neither theorem implies the other.
Nonlow$_2$ Bounding

Thm 3. [Karen Lange].
[Homogeneous Bounding Theorem] Given:

- A CD theory $T$;
- A homogeneous model $\mathcal{A} \models T$;
- A $0$-basis $X = S(\mathcal{A})$.
- A degree $d \leq 0'$ which is nonlow$_2$ ($d'' > 0''$).

Then there is a $d$-decidable copy $\mathcal{B} \cong \mathcal{A}$.

**Note.** Using Lange Homogeneous Low Basis Thm 1, strengthen to the $0'$-uniform case.

**Cor.** [Csima, Hirschfeldt, Knight, Soare] If $d \leq 0'$ is nonlow$_2$ then $d$ is prime bounding.
Escape Property

\[ D \leq_T \emptyset' \text{ nonlow}_2 \iff \]
\[ (\forall h \leq 0')(\exists f \leq_T D)(\exists \infty t)[ h(t) \leq f(t) ] \]
PIX: Show how the escape property guides one toward the MEF row. Use MEF not EEF.
Using Nonlow$_2$ to Search