Computability of Vaughtian Models:

Lecture 4: Degrees of Homogeneous Models

by

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MATHLOGAPS Conference
Leeds, U.K. Aug. 21–25, 2006

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Homogeneous Models

Def. $\mathcal{A}$ is *homogeneous* iff for all $\overline{a}$, and $\overline{b}$,

$$(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b}) \implies (\exists G \in \mathrm{Aut}(\mathcal{A}))[G(\overline{a}) = \overline{b}].$$

i.e., every finite elementary map $F(\overline{a}) = \overline{b}$ can be extended to an automorphism G of $\mathcal{A}$.

Def. For any model $\mathcal{A} \models T$ define the set of types *realized* in $\mathcal{A}$.

$$\mathbb{T}(\mathcal{A}) = \{p : p \in S(T) \ \ \& \ \ \mathcal{A} \text{ realizes } p\}.$$

Homogeneous Uniqueness Thm. Given a countable complete theory T and homogeneous models $\mathcal{A}$, $\mathcal{B}$ of T with $||\mathcal{A}|| = ||\mathcal{B}||$, then

$$\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}.$$

Proof of Homogeneous Uniqueness

Proof. Fix $\mathcal{A}$ and $\mathcal{B}$ be homogeneous, countable, with $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$. Suffices to define ω-back and forth $\mathcal{F}$ between $\mathcal{A}$ and $\mathcal{B}$. T is complete so $\mathcal{A} \equiv \mathcal{B}$. Add the empty map $\emptyset$ to $\mathcal{F}$.

Given any elementary map $f \in \mathcal{F}$, $f(\overline{a}) = \overline{b}$ and any $c \in A$ let p be the $(n+1)$-type of $(\overline{a}, c)$ in $\mathcal{A}$. There is some $(n+1)$-tuple $(\overline{b}\,', d\,')$ satisfying p in $\mathcal{B}$ **because** $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

Hence, $(\mathcal{B}, \overline{b}) \equiv (\mathcal{B}, \overline{b}\,')$. By **homogeneity** of $\mathcal{B}$ there is some $d \in B$ such that

$$(\mathcal{A}, \overline{a}, c) \ \equiv \ (\mathcal{B}, \overline{b}\,', d\,') \ \equiv \ (\mathcal{B}, \overline{b}, d).$$

Extend f to $g = f \cup \{(c, d)\}$ and add g to $\mathcal{F}$.

Spectrum of Homogeneous Models

Algebraically closed fields of characteristic 0

Baldwin-Lachlan sequence of countable models of ACF_0:

$$\overline{Q} \prec \overline{Q}[x_1] \prec \overline{Q}[x_1, x_2] \prec \ldots \quad \overline{Q}[x_i]_{i\in\omega}.$$

prime homogeneous saturated.

Spectrum of Ctble Homogeneous Models

$$\mathcal{A}_0 \qquad \ldots \quad \mathcal{A}_i \quad \ldots \qquad \mathcal{A}_\omega$$

$$S(\mathcal{A}_0) = S^P(T) \quad \subseteq S(\mathcal{A}_i) \quad \subseteq \quad S(\mathcal{A}_\omega) = S(T)$$

prime homogeneous saturated.

but the models are not linearly ordered.

Homogeneous Bounding Degrees

Def. A degree **d** is *homogeneous bounding* if every CD theory has a **d**-decidable homogeneous model.

Def. A degree **d** is a *Peano Arithmetic (PA) degree* if **d** is the degree of a complete extension of Peano Arithmetic.

Thm Csima, Harizanov, Hirschfeldt, Soare. A degree **d** is homogeneous bounding iff **d** is a PA degree.

PA Computes Homogeneous Model

Thm. Any countable theory T has a homogeneous model $\mathcal{A}$.

Build $\mathcal{A} = \cup \mathcal{A}_n$ and elementary chain. At level n list all finite partial elementary maps over $\mathcal{A}_n$ and in $\mathcal{A}_{n+1}$ add new constants to guarantee one apoint extension of each map.

This repeatedly uses Lindenbaum's Lemma (that every consistent set of sentences can be extended to a complete theory) which is equivalent to finding paths through trees.

Lindenbaum's Lemma can be carried out effectively in a degree **d** iff **d** is a PA degree.

Homogeneous Bounding is PA

Let U (V) be the set of Gödel numbers of sentences *provable (refutable)* from PA. Any separating set for U and V has a PA degree.

Key Idea. Build theory T s.t. if $\mathcal{A} \models T$ is homogeneous, then the atomic diagram of $D(\mathcal{A})$ can compute a separating set for U and V.

$\mathcal{L}(T)$ has infinitely many unary predicate symbols $\{P_i\}_{i\in\omega}$, infinitely many binary predicate symbols $\{R_i\}_{i\in\omega}$, a unary predicate symbol D, and a binary predicate symbol E.

Morley's Four Properties

Morley [1976, p. 236] noted:

P1. There is a decidable model $\mathcal{A}$.

P2. There is a computable listing of $\mathbb{T}(\mathcal{A})$.

P3. $\mathbb{T}(\mathcal{A})$ satisfies TAC (types all computable).

P4. The theory T is CD.

Morley noted the obvious:

$$P1 \quad \Longrightarrow \quad P2 \quad \Longrightarrow \quad P3 \quad \Longrightarrow P4.$$

$$P4 \quad \not\Longrightarrow \quad P3.$$

$$P3 \quad \not\Longrightarrow \quad P2.$$

Morley's Question

Def. Let $\mathcal{C} \subseteq S(T)$ be a set of types of a CD theory T If there exists some uniformly computable listing $X = \{p_j\}_{j \in \omega}$ of $\mathcal{C}$ we call X a $\mathbf{0}$-*basis* for $\mathcal{C}$.

Morley's Question. If T is a CD theory and $\mathcal{A}$ is a homogeneous model of T with a $\mathbf{0}$-basis X for $\mathbb{T}(\mathcal{B})$ does $\mathcal{A}$ have a decidable copy $\mathcal{B}$?

By the Homogeneous Uniqueness Thm this is equivalent to finding a decidable homogeneous model $\mathcal{B}$ of T with $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

Note. True for prime and saturated models.

Effective Extension Function (EEF)

Def [Effective Extension Function (EEF)] Let $\mathcal{A}$ be a homogeneous model of a CD theory T whose type spectrum $\mathbb{T}(\mathcal{A})$ has a $\mathbf{0}$-basis $X = \{p_i\}_{i \in \omega}$.

(i) A function f is an *extension function (EF)* for X if for every n,

- for every n-type $p_i(\overline{x}) \in X \cap S_n(T)$
- and every $(n+1)$-ary $\theta_j(\overline{x}, x_n) \in F_{n+1}(T)$ consistent with $p_i(\overline{x})$

$$p_i(\overline{x}) \cup \{\theta_j(\overline{x}, x_n)\} \subseteq p_{f(i,j)}(\overline{x}, x_n).$$

(ii) If f is also computable then f is an *effective extension function (EEF).*

Picture slide:

pix: Have one matrix, w/ p1 as 1-type on first row, and 2-ary fmls θ. Now move marker along 2-rows until it settles on right answer.

Picture of Extension Functions

Monotone Extension Function

Def. [**Monotone Function on** X] A function $f(i)$ on X is a *monotone function* on X if there is a computable function $\widehat{f}(i,s)$ such that,

(i) $f(i) \;=\; lim_s\, \widehat{f}(i,s)$, and

(ii) $p_{\,\widehat{f}(i,s)} \restriction s \;\subseteq\; p_{\,\widehat{f}(i,s+1)} \restriction (s+1)$.

In this case we write $f(i) = mlim_s\; \widehat{f}(i,s)$.

IDEA. We build a computable type $q = \cup_s\, p_{\widehat{f}(i,j,s)}{\restriction} s$ as the union of a monotone sequence $\{p_{\widehat{f}(i,j,s)}{\restriction} s\}$.

Thm. If $X = \{p_i\}_{i\in\omega}$ and $Y = \{q_i\}_{i\in\omega}$ are $\mathbf{0}$-bases for $\mathbb{T}(\mathcal{A})$ there is a monotonic function g on X such that $p_{g(i)} = q_i$.

EEF and Decidable Copies

Positive Thm. [Goncharov, Peretyatkin].
Let T be a CD theory and $\mathcal{A} \models T$ homogeneous.
TFAE:

(i) $\mathcal{A}$ has a decidable copy $\mathcal{B}$.

(ii) *Every* 0-basis $X = S(\mathcal{A})$ has MEF.

(iii) *Some* 0-basis for $S(\mathcal{A})$ has MEF.

(iv) Some 0-basis for $S(\mathcal{A})$ has EEF.

Proof. $(i) \Longrightarrow (iv)$, obvious.

$(ii) \iff (iii) \iff (iv)$, easy.

$(iv) \Rightarrow (i)$, main import of the theorem.

Corollaries: Prime and Saturated

Application Thm. Let T be CD and $\mathcal{A} \models T$ have 0-basis $X = S(\mathcal{A})$. If $\mathcal{A}$ is either:

- prime, or
- saturated,

then X has MEF. (Hence, $\exists$ decidable $\mathcal{B} \cong \mathcal{A}$.)

Coroll [Harrington, Goncharov-Nurtazin]. If T is a complete atomic decidable (CAD) theory and $S^P(T)$ has a 0-basis then T has a decidable prime model.

Coroll [Morley, Millar]. If T is a complete decidable (CD) theory and $S(T)$ has a 0-basis, then T has a decidable saturated model.

PIX: Ease to see prime case by movable marker. Satur. easy also.

Picture of Prime Model and MEF

Negative Thm.
[**Goncharov, Peretyatkin, Millar**].
There exists:

- a CD theory T,
- a homogeneous model $\mathcal{A} \models T$,
- a 0-basis $X = S(\mathcal{A})$,

with *no* decidable copy $\mathcal{B} \cong \mathcal{A}$.

Proof.
Construct 0-basis $X = S(\mathcal{A})$ with no MEF.

Pix. Push opponent’s MEF fn off to infty.

Picture of Negative Theorem

Homogeneity Conditions

Existence Thm for Ctble Homogeneous Models [Goncharov, Peretyat'kin].
Given CD theory T and ctble $S \subseteq S(T)$.

$$(\exists \text{ homogeneous } \mathcal{A} \vDash T)\ [\ S(\mathcal{A}) = S\ ]$$

$$\Longleftrightarrow$$

1. S is closed under taking subtypes, and
2. S is closed under permutations of variables,
3. (EP) If $p(x_1, ..., x_n) \in S$ and $\theta(x_1, ..., x_{n+1})$ are consistent, then there exists an $(n+1)$-type $q \in S$ such that $p \cup \{\theta\} \subseteq q$, and
4. (TAP) For any two types $p_1(\bar{x}, y)$, $p_2(\bar{x}, z) \in S$ such that $p_1 \downarrow \bar{x} = p_2 \downarrow \bar{x}$, there exists a type $q(\bar{x}, y, z)$ containing p_1 and p_2.

Degrees of Homogeneous Models

Thm 1. [Karen Lange]
[**Homogeneous Low Basis Thm**]. Given:

- A CD theory T;
- A homogeneous model $\mathcal{A} \vDash T$;
- A $\mathbf{0}'$-basis $X = S(\mathcal{A})$.

Then there is a copy $\mathcal{B} \cong \mathcal{A}$ which is *low*.
(Namely, $D^e(\mathcal{B})' \equiv_{\mathrm{T}} 0'$.)

Coroll. [**Prime Low Basis Thm, Csima**]
Every complete atomic decidable (CAD) theory T has a a low prime model $\mathcal{A}$.

Prf. If T is CAD, then *any* prime model $\mathcal{A} \vDash T$ has a $0'$-basis $X = S(A) = S^P(T)$.

$S(T)$ Types All Computable (TAC)

Thm 2. [**Karen Lange**].
[**Homogeneous Full Basis Theorem**]
Let T be a CD theory with types all computable (TAC). Let homogeneous $\mathcal{A} \models T$ have a $\mathbf{0}$-basis. Then

$$\{\mathbf{d} : \mathbf{0} < \mathbf{d}\} \subseteq \{ \deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A} \}.$$

Note. Like the Csima-Hirschfeldt Full Basis Thm for prime models of a CAD theory T with TAC. Neither theorem implies the other.

Nonlow$_2$ Bounding

Thm 3. [**Karen Lange**].
[**Homogeneous Bounding Theorem**] Given:

- A CD theory T;
- A homogeneous model $\mathcal{A} \models T$;
- A $\mathbf{0}$-basis $X = S(\mathcal{A})$.
- A degree $\mathbf{d} \leq \mathbf{0}'$ which is *nonlow*$_2$ ($\mathbf{d}'' > \mathbf{0}''$).

Then there is a $\mathbf{d}$-decidable copy $\mathcal{B} \cong \mathcal{A}$.

Note. Using Lange Homogeneous Low Basis Thm 1, strengthen to the $\mathbf{0}'$-uniform case.

Cor. [**Csima, Hirschfeldt, Knight, Soare**] If $\mathbf{d} \leq \mathbf{0}'$ is *nonlow*$_2$ then $\mathbf{d}$ is prime bounding.

Escape Property

Escape Property

$$D \leq_{\mathrm{T}} \emptyset' \ nonlow_2 \iff$$

$$(\forall h \leq \mathbf{0}')(\exists f \leq_{\mathrm{T}} D)(\exists^{\infty} t)[\ h(t) \leq f(t)\ ]$$

PIX: Show how the escape property guides one toward the MEF row. Use MEF not EEF.

Using Nonlow$_2$ to Search