Computability of Vaughtian Models:

Lecture 3: Degrees of Saturated Models

by

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Homogeneous Models

Def. $\mathcal{A}$ is *homogeneous* iff for all $\overline{a}$, and $\overline{b}$,

$$(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b}) \implies (\exists G \in \mathrm{Aut}(\mathcal{A}))[G(\overline{a}) = \overline{b}].$$

i.e., every finite elementary map $F(\overline{a}) = \overline{b}$ can be extended to an automorphism G of $\mathcal{A}$.

Def. For any model $\mathcal{A} \models T$ define the set of types *realized* in $\mathcal{A}$.

$$\mathbb{T}(\mathcal{A}) = \{p : p \in S(T) \ \ \& \ \ \mathcal{A} \text{ realizes } p\}.$$

Homogeneous Uniqueness Thm. Given a countable complete theory T and homogeneous models $\mathcal{A}$, $\mathcal{B}$ of T with $||\mathcal{A}|| = ||\mathcal{B}||$, then

$$\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \implies \mathcal{A} \cong \mathcal{B}.$$

Proof of Homogeneous Uniqueness

Proof. Fix $\mathcal{A}$ and $\mathcal{B}$ be homogeneous, countable, with $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$. Suffices to define ω-back and forth $\mathcal{F}$ between $\mathcal{A}$ and $\mathcal{B}$. T is complete so $\mathcal{A} \equiv \mathcal{B}$. Add the empty map $\emptyset$ to $\mathcal{F}$.

Given any elementary map $f \in \mathcal{F}$, $f(\overline{a}) = \overline{b}$ and any $c \in A$ let p be the $(n+1)$-type of $(\overline{a}, c)$ in $\mathcal{A}$. There is some $(n+1)$-tuple $(\overline{b}\,', d\,')$ satisfying p in $\mathcal{B}$ **because** $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

Hence, $(\mathcal{B}, \overline{b}) \equiv (\mathcal{B}, \overline{b}\,')$. By **homogeneity** of $\mathcal{B}$ there is some $d \in B$ such that

$$(\mathcal{A}, \overline{a}, c) \;\equiv\; (\mathcal{B}, \overline{b}\,', d\,') \;\equiv\; (\mathcal{B}, \overline{b}, d).$$

Extend f to $g = f \cup \{(c, d)\}$ and add g to $\mathcal{F}$.

Saturated Models

Def. Let T be a countable complete theory, $\mathcal{A} \models T$ countable.

(i) $\mathcal{A}$ is *saturated* if every 1-type $p(\overline{a}, x)$ over a finite set of elements $\overline{a} \in A$ is realized in $\mathcal{A}$.

(ii) $\mathcal{A}$ is *weakly saturated* if $\mathbb{T}(\mathcal{A}) = S(T)$, *i.e.*, every *pure* type $p \in S(T)$ is realized in $\mathcal{A}$.

Thm (Vaught). If T is complete and $\mathcal{A} \models T$ is countable then TFAE:

(i) $\mathcal{A}$ is saturated.

(ii) $\mathcal{A}$ is weakly saturated and homogeneous.

Countable Saturated Models

Examples

Example 1. Let $T = DLO$ dense linear orderings w/o endpoints. Let $\mathcal{A} = (A, <)$ where A is the rationals but $\mathcal{A}$ has no constants names. Then $\mathcal{A}$ is the countable saturated model of T.

Example 2. Let $T = ACF_0$ algebraically closed fields of characteristic zero. Let

$$\mathcal{A} = (\overline{Q}, \{x_i\}_{i\in\omega})$$

the algebraic numbers $\overline{Q}$ with infinitely many transcendentals x_i adjoined.

Example 3. Any complete theory formed from a tree with every point having Cantor Bendixson rank.

Uniqueness of Prime, Saturated

Coroll. T a complete theory, $\mathcal{A}$, $\mathcal{B}$ countable models of T.

(i) $\mathcal{A}$ and $\mathcal{B}$ prime $\implies \quad \mathcal{A} \cong \mathcal{B}$.

(ii) $\mathcal{A}$ and $\mathcal{B}$ saturated $\implies \quad \mathcal{A} \cong \mathcal{B}$.

Proof. Both prime, saturated are homogeneous.

(i) $\mathcal{A}$ prime $\implies \quad \mathbb{T}(\mathcal{A}) = S^P(T)$.

(ii) $\mathcal{A}$ saturated $\implies \quad \mathbb{T}(\mathcal{A}) = S(T)$.

Saturated Existence Theorem

Saturated Model Existence Thm (Vaught) A theory T has a countable saturated model iff $S(T)$ is countable.

Proof. If T has a countable saturated model $\mathcal{A}$, then $\mathbb{T}(\mathcal{A}) = S(T)$ is countable.

Suppose $S(T) = \{p_i\}_{i \in \omega}$. We build $\mathcal{A} = \cup_s \mathcal{A}_s$ the union of an elementary chain. Given $\mathcal{A}_s$ and a type $p(\overline{x})$ with finitely many new constants $\overline{c} \in C$ we realize type p with fresh constants $\overline{d} \in C$ by adding $\theta(\overline{d})$ for all $\theta(\overline{x}) \in p$ and then taking a model $\mathcal{A}_{s+1}$.

Decidable Saturated Models

T**:** a complete decidable (CD) theory with types all computable (TAC).

Thm (Morley, Millar) Saturated Criterion
TFAE:

(i) T has a decidable saturated model $\mathcal{A}$.

(ii) There is a computable listing of $S(T)$, *i.e.*, a $\mathbf{0}$-basis for $S(T)$.

Proof. (i) $\Longrightarrow$ (ii). Let $\mathcal{A}$ be decidable,

$$\mathbb{T}(\mathcal{A}) \; = \; \{ \; p(\overline{a}) \; : \; \overline{a} \in \mathcal{A} \; \}.$$

(ii) $\Longrightarrow$ (i). (Same problem as in prime case.) Let $\{p_i\}_{i \in \omega}$ be effective listing of $S(T)$.

Proof Cont'd

Choose a 1-type p_1, put $p(c_1)$ into $D^e(\mathcal{A})$. For (c_1, c_2) we need 2-type p_2 consistent with p_1, then put $p_2(c_1, c_2)$ into $D^e(\mathcal{A})$. But consistency of p_1 and p_2 is a Π_1 property, decidable in $\mathbf{0}'$. ...

(Return to this later with homogen. case.)

Coroll. Let T be a CD theory with TAC, and $\mathcal{A}$ a prime or saturated model of T. Then TFAE:

(i) $\mathcal{A}$ has a decidable copy $\mathcal{B}$.

(ii) There is a computable enumeration ($\mathbf{0}$-basis) for $\mathbb{T}(\mathcal{A})$.

Degrees of Saturated Models

Thm. For theory T CD + TAC there is a saturated model $\mathcal{A} \models T$ s.t. $\mathcal{A} \leq_{\mathrm{T}} \emptyset'$.

Def. (i) If $f(x, y)$ is a binary function then

$$f_y \text{ denotes } \lambda x\,[\, f(x, y)\,].$$

We view $\lambda x, y\,[\, f(x, y)\,]$ as specifying a *matrix* whose rows with entry $f(x, y)$ on the location (x, y) under by usual cooordinates. For vertical coordinate $y \in \omega$ we view $\lambda x\,[\, f_y(x)\,]$ as the y^{th} *row* (viewed horizontally).

(ii) If $\mathcal{C}$ is a class of (unary) functions and $\mathbf{d}$ is a degree, $\mathcal{C}$ is called $\mathbf{d}$-*unifo* if there is a binary function f of degree $\leq \mathbf{d}$ such that

$$\mathcal{C} = \{\, f_y \,\}_{y \in \omega}.$$

Uniform numerations

Thm Jockusch. If $\mathbf{d}$ is any degree, then TFAE:

(i) $\mathbf{d}' \geq \mathbf{0}''$ (i.e., $\mathbf{d}$ is high);

(ii) the computable functions are $\mathbf{d}$-unorm;

Proof. (i) $\Longrightarrow$ (ii). By Martin's Thm choose a dominant function $g \leq \mathbf{d}$. Define p.c. fn

$$c_e(x) = (\mu s)(\forall y \leq x)[\ \varphi_{e,s}(x)\downarrow\ ].$$

$$\varphi_e \text{ total } \Longrightarrow c_e \text{ total }.$$

$$c_e \text{ total } \Longrightarrow (\exists x_e)(\forall x \geq x_e)[\ c_e(x) < g(x)\ ].$$

Define $f(\langle e, i\rangle, x) = \varphi_{e,i+g(x)}(x)$ if $\varphi_{e,i+g(y)}(y)\downarrow$ for all $y \leq x$ and $f(\langle e, i\rangle, x) = 0$ otherwise. Either $f_{\langle e,i\rangle} = \varphi_e$ total, or $f_{\langle e,i\rangle}$ is finitely nonzero. So $f_{\langle e,i\rangle}$ is computable. If φ_e is total then $g(x)$ dominates $c_e(x)$, so $\varphi_e = f_{\langle e,i\rangle}$ for some i.

Enumerating a PAC Tree

Thm. Let $\mathcal{T}$ be an extendible PAC tree, and $\mathbf{d}$ a high degree. There is a $\mathbf{d}$-uniform listing of $[\,\mathcal{T}\,]$.

Proof. Let $g(x, y)$, $g \leq \mathbf{d}$ be a matrix whose rows $\{g_y\}_{y \in \omega}$ = all total computable functions.

Build $h \leq \mathbf{d}$ such that the rows $\{h_y\}_{y \in \omega} = [\,\mathcal{T}\,]$.

Let $h_i = g_i$ if $g_i \in [\mathcal{T}]$ and $h_i \in [\mathcal{T}]$ o.w.

High Saturated Models

Coroll. If T is a CD + TAC theory and $\mathbf{d}$ is a high degree then T has a $\mathbf{d}$-decidable saturated model $\mathcal{A}$.

1. Use Jockusch to get a $\mathbf{d}$-uniform enumeration of $S(T)$.
2. Use Morley-Millar to get $\mathbf{d}$-decidable saturated model.

Thm (Jockusch). If $\mathbf{d}$ is a degree of Peano arithmetic (PA) then there is a $\mathbf{d}$-decidable saturated model.

Proof. Any PA degree can subenumerate the computable sets and hence every PAC Π^0_1-class.

No Decidable Saturated Model

Thm(Millar). There is a CD + TAC theory T with no decidable saturated model.

Proof.

T Player plays: CD + TAC theory T.

$\mathcal{A}$ Player plays: a computable matrix g by

$$\varphi_e(x, y) = g(x, y)$$

The matrix g has rows $g_y = \lambda x[\, g(x, y)\,]$.
T *wins* if $S(T) \neq \{g_y\}_{y \in \omega}$, and $\mathcal{A}$ wins otherwise.

The T-strategy is to build an extendible computable PAC tree and to play above node $1^e {}^\frown 0$ to ensure that if $\varphi_e = g$ then

$$(\exists f \in S(T))\, [\, f \notin \{g_y\}_{y \in \omega}\,].$$

(See picture of the strategy to meet $R_{e,y}$.)

T with No Low Saturated Model

Any CAD theory T has a *low* prime model. In contrast for *saturated* models,

Thm (Ken Harris). There is a CD + TAC theory T with no *low* saturated model.

Saturated Bounding Degrees

Def. A degree $\mathbf{d}$ is *saturated bounding* if for every CD+TAC theory T there is a saturated model $\mathcal{A}$ of T which is $\mathbf{d}$-decidable.

Blue. **Positive**: Can always get a saturated model of degree $\mathbf{d}$. e.g. $\mathbf{d} = \mathbf{0}'$ or high $\mathbf{d}' = \mathbf{0}''$.

Red. Negative: Cannot always get sat. model of degree $\mathbf{d}$, e.g. $\mathbf{d} = \mathbf{0}$ or $\mathbf{d}$ low ($\mathbf{d}' = \mathbf{0}'$).

(See diagram of oval between $\mathbf{0}'$ and $\mathbf{0}$.)

Uniform Escape Property

Def. A degree $\mathbf{d}$ has the *Uniform Escape Property* if there is an $h \leq_{\mathrm{T}} \mathbf{0}$ such that

$$(\forall e)[\Phi_e^{\mathbf{d}} \text{ total } \implies (\exists^\infty x)[\Phi_e^{\mathbf{d}}(x) \leq \Phi_{h(e)}(x)]].$$

Thm. For *c.e.* degrees $\mathbf{d}$, TFAE:

(a) $\mathbf{d}$ is *low*.

(b) $\mathbf{d}$ has the Uniform Escape Property.

Extending Negative Results

Def. A degree $\mathbf{d}$ is *low$_n$* if $\mathbf{d}^{(n)} = 0^{(n)}$.

Thm (K. Harris) For $n \geq 1$ TFAE:

(i) A is low$_n$.

(ii) A has n-UEP.

Def. A refinement of n-UEP is the *aligned escape property (AEP).*

Thm. All low$_n$ c.e. degrees have AEP.

Thm. A has n-AEP $\implies$ A not saturated bounding.

Coroll. No low$_n$ c.e. degree is saturated bounding.

Low$_n$ Degrees and Escape Functions

There is a hierarchy of properties characterized by less effective procedures, **Uniform Escape Property** n**-UEP**, starting with (1-UEP)=(UEP), such that

Thm. For all degrees $\mathbf{d}$ and all $n \geq 1$ TFAE:

(i) $\mathbf{d}$ is $\mathbf{L}_n$ $(\mathbf{d}^{(n)} = \mathbf{0}^{(n)})$.

(ii) $\mathbf{d}$ has (n-UEP).

Infinitary Quantifiers

$(\forall^{\infty} x)$: *For almost every* x.
Reduces to $\exists\forall$ and behaves like $\forall$.

$(\exists^{\infty} x)$: *There exists infinitely many* x.
Reduces to $\forall\exists$ and behaves like $\exists$.

n-Uniform Escape Property

Def. Degree $\mathbf{d}$ has the *n-Uniform Escape Property* (n-UEP) if for any set $A \in \mathbf{d}$:

There are uniformly enumerable (u.e.) families of partial computable functions $\lambda e[\, h_{e,\overline{y}}\,],\ \overline{y} \in \omega\,]$

such that for any u.e. family $\{\Phi^A_{e,\overline{y}}\}_{\overline{y}\in\omega}$ with

$$(Q_1 y_{n-1})(Q_2 y_{n-2})\ldots\ [\Phi^A_{e,\overline{y}} \text{ total }] \quad \Longrightarrow$$

$$(Q_1 y_{n-1})(Q_2 y_{n-2})\ldots\ [\ h_{e,\overline{y}} \text{ total } \ \& \ \text{ escapes } \Phi^A_{e,\overline{y}}\ ]$$

where $Q_1, Q_2 \in \{\exists^\infty,\ \forall^\infty\}$ by

- For *odd* n: alternate $\exists^\infty\forall^\infty$
- For *even* n: alternate $\forall^\infty\exists^\infty$