Computability of Vaughtian Models:

Lecture 2: Degrees Bounding Prime Models

by

Robert I. Soare

The Unversity of Chicago

MATHLOGAPS Conference
Leeds, U.K. Aug. 21–25, 2006

WEB SITE:

www.math.uchicago.edu/∼soare/ or
www.cs.uchicago.edu/∼soare/

Types All Computable

Def. Tree $\mathcal{T} \subset 2^{<\omega}$ is a *PAC tree* if it is a computable extendible tree and every path in $[\mathcal{T}]$ is computable. (*PAC* means *paths all computable.*)
(*TAC* means *types all computable.*)

Thm (Hirschfeldt). If $\mathcal{T}$ be an extendible PAC tree and $D >_{\mathrm{T}} \emptyset$ there is a D-computable listing of the isolated paths in $[\,\mathcal{T}\,]$.

Coroll. If T is a CAD + TAC theory and
$D >_{\mathrm{T}} \emptyset$ then T has a D-decidable prime model.

Coroll. If $\mathbf{0} \notin dgSp(\mathcal{A})$, $\mathcal{A}$ prime, then

$$dgSp(\mathcal{A}) \;=\; \{\, \mathbf{d} : \mathbf{d} > \mathbf{0} \,\}.$$

Coroll (Slaman, Wehner). There is a structure with presentations of every nonzero degree but no computable presentation.

Proof of Hirschfeldt PAC Thm

Let $\{\sigma_i\}_{i\in\omega}$ be effective list of $\mathcal{T}$. Define

$$[\,\mathcal{T}\,]^P = \{f : f \in [\,T\,] \;\;\&\;\; f \text{ principal (isolated)}\}.$$

By Harrington (iii) we build $g(\sigma, y)$, where $g \leq_{\mathrm{T}} D$ s.t.

$$(\forall \sigma \in \mathcal{T})\; [\;\; g_\sigma \in \;\; [\,\mathcal{T}\,]^P \;\;],$$

where $g_\sigma \;=\; \lambda y\; [\; g(\sigma, y) \;]$.

Problem: To guarantee that each g_σ is isolated.

Construction of g_σ.

Begin with $g^0 = \sigma$. Given $g^s = \tau$ and both $\tau^j \in \mathcal{T}$, $j = 0, 1$ and this is the k^{th} splitting bertween g^0 and τ. Define $g^{s+1} = \tau^{D(k)}$.

Path g_σ is Isolated

Lemma. g_σ is isolated.

Proof. Suppose $g_\sigma \in [\mathcal{T}\,] \; - \; [\mathcal{T}\,]^P$. Then

$$D \leq_{\mathrm{T}} \; g_\sigma \; \oplus \; \mathcal{T}.$$

$$\therefore \quad D \text{ is computable } .$$

High Degrees and Domination

Def. (i) Function g *dominates* f ($f <^* g$) if

$$(\forall^\infty x)\,[\, f(x) < g(x) \,].$$

(ii) f *escapes* (domination by) g if $f \not<^* g$, *i.e.*,

$$(\exists^\infty x)\,[\, g(x) \leq f(x) \,].$$

(iii) f is *dominant* if f dominates every (total) computable function.

Def. A degree $d \leq \mathbf{0}'$ is *high* if $\mathbf{d}' = \mathbf{0}'$

Thm (Martin). A degree $\mathbf{d}$ is high iff

$$\exists \text{ dominant } g \leq_{\mathrm{T}} \mathbf{d}.$$

Bounding Prime Models

(P0) *The escape property.*
$(\forall g \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^{\infty} x)\,[\ g(x) \leq f(x)\ ]$,

$(\exists^{\infty})$ denotes “there exist infinitely many”.

(P1) *The nonlow$_2$ property.*
X is not low$_2$ (*i.e.*, $X'' >_{\mathrm{T}} 0''$).

(P2) *The prime bounding property.*
X is prime bounding,
(i.e. every CAD theory T has an X-decidable prime model.)

Thm (Csima, Hirschfeldt, Knight, Soare).
For $X \leq_{\mathrm{T}} 0'$

$$(P0) \quad \Longleftrightarrow \quad (P1) \quad \Longleftrightarrow \quad (P2).$$

Properties (P3) – (P8)

(P3) *The isolated path property.* For every computable tree $\mathcal{T} \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall \sigma \in \mathcal{T})\,[\, g_\sigma \in [\mathcal{T}_\sigma] \;\;\&\;\; g_\sigma \text{ is isolated}\,].$$

(P4) *The tree property.* For every computable extendible tree $\mathcal{T} \subseteq 2^{<\omega}$, and uniformly Δ^0_2 sequence of subsets $\{S_i\}_{i\in\omega}$ dense in $\mathcal{T}$, there exists $g \leq_{\mathrm{T}} X$ for all $\sigma \in \mathcal{T}$, $g_\sigma = \lambda y\,[g(\sigma, y)]$ is a path extending σ and hitting each S_i, *i.e.*,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall \sigma \in \mathcal{T})\,[\,\sigma \subset g_\sigma$$
$$\&\;\; (\forall i)(\exists \tau \in S_i)[\tau \subset g_\sigma \in [\mathcal{T}]\,]\,].$$

Topology

Property (P4) has a topological interpretation in the Cantor Space 2^ω. Recall in Cantor Space the **basic open sets** are

$$U_\sigma = \{f : f \in \ 2^\omega \quad \& \ \ \sigma \ \subset f\}.$$

and **open sets** are

$$U_S = \bigcup \{\ U_\sigma : \sigma \in S\ \}.$$

Hence, **(P4) says** that for every $\sigma \in \mathcal{T}$, the path $g_\sigma \in [\mathcal{T}]$ extends σ and lies in **every dense open** set U_{S_i}. This says for the Δ^0_2 family $\mathcal{G} = \{S_i\}_{i\in\omega}$ that X can compute a $\mathcal{G}$-generic path g. A special case is that X computes a 1-generic set.

Omitting Types

(P5) *The omitting types property.* For any complete decidable theory T and any uniformly Δ^0_2 family of sets of formulas $\{\Gamma_j(\overline{x}_j)\}_{j\in\omega}$, all nonprincipal with respect to T, there is an X-decidable model of T omitting all $\Gamma_j(\overline{x}_j)$.

Def. A set $S \subset \omega$ is *X-monotonic* if there is a function $g \leq_T X$ such that for every x, $g(x,y)$ is nondecreasing in y, with limit $\widehat{g}(x) = \lim_y g(x,y)$, $\widehat{g}(x) \geq x$, and $\widehat{g}(x) \in S$.

(P6) *The monotonic property.* Every infinite Δ^0_2 set S, is X-monotonic, *i.e.*,

$$(\exists g \leq_{\mathrm{T}} X)\,(\forall x)\,(\forall y)$$

$$[x \leq g_x(y) \leq g_x(y+1) \quad \& \quad \lim_y g(x,y)\!\downarrow\in S].$$

Algebraic Properties

An *equivalence structure* is a structure of the form $\mathcal{A} = (A, E)$, where E is an equivalence relation on A.

(P7) *The equivalence structure property.* For any Δ^0_2 set $S \subseteq \omega - \{0\}$, there is an X-computable equivalence structure with one class of size n for each $n \in S$, and no other classes.

(P8) *The Abelian p-group property.* For any infinite Δ^0_2 set $S \subseteq \omega - \{0\}$, there is an X-computable reduced Abelian p-group $\mathcal{G}$, of length ω, and with $u_n(\mathcal{G}) \leq 1$ for all n, such that $S(\mathcal{G}) = S$.

Escape (P0) Implies Isolated (P3)

Assume X satisfies the **escape property** (P0),

(1) $(\forall h \leq_{\mathrm{T}} 0')\,(\exists f \leq_{\mathrm{T}} X)\,(\exists^{\infty} x)\,[\ h(x) \leq f(x)\ ]$.

Let $\mathcal{T} \subseteq 2^{<\omega}$ be a computable extendible tree with isolated paths dense. Define $g(\sigma, s) \leq_{\mathrm{T}} X$,

$$(\forall \sigma \in \mathcal{T})[\ \sigma \subset g_{\sigma} \in [\mathcal{T}]^{P}\ ].$$

Let S be the set of *atoms* of $\mathcal{T}$, *i.e.*, nodes σ with a unique extension $f \in [\mathcal{T}_{\sigma}]$. Since S is Π^0_1 and hence Δ^0_2, there is a computable sequence $\{S_s\}_{s\in\omega}$ such that $S(x) = \lim_s S_s(x)$ for all x. Assume $\forall \tau \in \mathcal{T}$, $\forall s$, S_s contains some $\rho \supseteq \tau$.

For every $z \in \mathcal{T}$ define the *target,*

$$y_z = (\mu y)[z \subset y\ \ \&\ \ y \in S], \qquad \text{and}$$

$$y^s_z = (\mu y)[z \subset y\ \ \&\ \ y \in S_s].$$

Using the Escape Function

Def. Define fn $h \leq_{\mathrm{T}} 0'$.

$$h(n) = (\mu s)\,(\forall z)_{|z| \leq n}\,(\forall w \leq y_z^s)$$

$$(\forall t \geq s)[\,S_t(w) = S_s(w) = S(w)\,].$$

(Note h total because $(\forall n)$ $(\exists^{<\mathrm{inf}} z)[|z| \leq n]$. $(\forall z)(\forall s)$ the apparent target y_z^s stabilizes using $S(x) = \lim_s S_s(x)$.)

By the escape property (P0) in (1),

$$(\exists f \leq_{\mathrm{T}} X)\,(\exists T \text{ infinite })\,(\forall t \in T\,)\,[\ h(t) \leq f(t)\ ].$$

f is monotonic. Call T the set of *true stages*.

True Stages

Speed up to X-computable sequence $\widehat{S}_s = S_{f(s)}$.
Define $\widehat{y}_z^s = y_z^{f(s)}$ X-computable in z and s.

Note. Any *apparent* target $\widehat{y}_z^t$ at a true stage $t \in T$ is the *true* target y_z, *i.e.*,

$$(\forall t \in T)\, (\forall z)_{|z| \leq t}\, (\forall v \geq t)[\, \widehat{y}_z^t = \widehat{y}_z^v = y_z\,].$$

For $s \leq |x|$ define $g(x, s) = x \restriction s$. Fix $s \geq |x|$ and assume we are given $g(x, s)$ with $|g(x, s)| = s$. Define $g(x, s+1) = \widehat{y}_{g(x,s)}^s \restriction (s+1)$.

$(\forall s > |x|)[x \subset g(x, s) \subset g(x, s{+}1) \;\; \& \;\; |g(x, s)| = s]$.

Key Point

KEY POINT.
If $t \in T$ and $y = \widehat{y}^t_{g(x,t)}$, then for every s with $t < s < v = |y|$, we have $\widehat{y}^s_{g(x,s)} = y$, because y will be the most attractive target for $g(x, s)$ since no elements $w \leq y$ enter or leave S after stage t.

Hence, if $t \in T$, then the sequence $\{\, g(x, s) \, : \, t < s \leq v \,\}$ marches **inexorably** from $g(x, t)$ toward y until hitting it at stage v, even though the intermediate stages s with $t < s < v$, need *not* be in T. Hence, $g_x \in U_S$, and so g_x is an isolated path.

Theories and Π^0_1-Classes

Def. $\mathcal{C} \subseteq 2^\omega$ is a Π^0_1-*class* if there is a computable relation $R(x)$ such that

$$\mathcal{C} = \{f : (\forall x)\, R(f(x))\}.$$

or equivalently a computable (not necessarily extendible) tree $\mathcal{T} \subset 2^{<\omega}$ s.t.

$$\mathcal{C} = [\mathcal{T}].$$

Thm. If T is an axiomatizable theory, then the class of complete extensions is Π^0_1. Put θ_α on $\mathcal{T}$ if $|\alpha| = s$ and

$$(\forall \beta \subseteq \alpha)[T_s \quad \nvdash \ \neg\theta_\beta\ ],$$

i.e., if θ_α seems consis with T after s steps.

PA and Low Basis Theorem

Def. **Peano Arithmetic (PA)** is the first order theory of arithmetic with induction.

Cor. There is a computable tree $\mathcal{T}_P \subseteq 2^{\langle\omega}$ such that

$$[\mathcal{T}_P] \; \{ \, f \; : \; f \text{ a complete extension of PA } \}.$$

Thm (Jockusch-Soare). If $\mathcal{T} \subseteq 2^{<\omega}$ and $[\mathcal{T}] \neq \emptyset$, then

$$(\exists f \text{ low })[\, f \in [\mathcal{T}] \,].$$

Cor. $[\mathcal{T}_P]$ contains a low complete extension of PA.

PA Climbs Trees

Thm. Let $[\mathcal{T}]$ be a nonempty Π^0_1 class, and g any complete extension of PA. Then

$$(\exists f \leq_{\mathrm{T}} g)[\, f \in [\,\mathcal{T}\,]\,].$$

Proof. Ask Rosser-type question about climbing a tree.

Cor. ($\exists$ low complete extension g of PA)
($\forall$ Π^0_1 nonempty class $[\mathcal{T}]$)

$$(\exists f \leq_{\mathrm{T}} g)[\, f \in [\,\mathcal{T}\,]\,].$$

Cor. The Low Basis Thm does not imply low prime models.

Def.

Def. **Thm.**

Def.

Def. **Thm.**

Def.

Def. **Thm.**