Computability of Vaughtian Models:

Lecture 2: Degrees Bounding Prime Models

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**Def.** Tree $T \subset 2^{<\omega}$ is a **PAC tree** if it is a computable extendible tree and every path in $[T]$ is computable. (*PAC* means *paths all computable.*)  
(*TAC* means *types all computable.*)

**Thm (Hirschfeldt).** If $T$ be an extendible PAC tree and $D >_T \emptyset$ there is a $D$-computable listing of the isolated paths in $[T]$.

**Coroll.** If $T$ is a CAD + TAC theory and $D >_T \emptyset$ then $T$ has a $D$-decidable prime model.

**Coroll.** If $0 \notin dgSp(A)$, $A$ prime, then

$$dgSp(A) = \{ d : d > 0 \}.$$ 

**Coroll (Slaman, Wehner).** There is a structure with presentations of every nonzero degree but no computable presentation.
Proof of Hirschfeldt PAC Thm

Let \( \{ \sigma_i \}_{i \in \omega} \) be effective list of \( T \). Define

\[ [T]^P = \{ f : f \in [T] \ & \ f \text{ principal (isolated)} \} \]

By Harrington (iii) we build \( g(\sigma, y) \), where \( g \leq_T D \) s.t.

\[ (\forall \sigma \in T) \ [ g_\sigma \in [T]^P ] \]

where \( g_\sigma = \lambda y \ [ g(\sigma, y) ] \).

Problem: To guarantee that each \( g_\sigma \) is isolated.

Construction of \( g_\sigma \).

Begin with \( g^0 = \sigma \). Given \( g^s = \tau \) and both \( \tau^j \in T, j = 0, 1 \) and this is the \( k^{th} \) splitting between \( g^0 \) and \( \tau \). Define \( g^{s+1} = \tau^{D(k)} \).
Lemma. \( g_\sigma \) is isolated.

Proof. Suppose \( g_\sigma \in [T] - [T]^P \). Then

\[
D \leq_T g_\sigma \oplus T.
\]

\[\therefore D \text{ is computable}.\]
Def.  (i) Function $g$ dominates $f$ ($f <^* g$) if
\[(\forall^\infty x) \ [ f(x) < g(x) ]. \]

(ii) $f$ escapes (domination by) $g$ if $f \not<^* g$, i.e.,
\[(\exists^\infty x) \ [ g(x) \leq f(x) ]. \]

(iii) $f$ is dominant if $f$ dominates every (total) computable function.

Def.  A degree $d \leq 0'$ is high if $d' = 0'$

Thm (Martin).  A degree $d$ is high iff
\[\exists \text{ dominant } g \leq_T d.\]
Bounding Prime Models

(P0) The escape property.
\( (\forall g \leq T 0') (\exists f \leq T X) (\exists^\infty x) [ g(x) \leq f(x) ] \),
(\(\exists^\infty\)) denotes “there exist infinitely many”.

(P1) The nonlow\(_2\) property.
\( X \) is not low\(_2\) (i.e., \( X'' > T 0'' \)).

(P2) The prime bounding property.
\( X \) is prime bounding,
(i.e. every CAD theory \( T \) has an \( X \)-decidable prime model.)

Thm (Csima, Hirschfeldt, Knight, Soare).
For \( X \leq_T 0' \)
\[ (P0) \iff (P1) \iff (P2) \]
Properties (P3) – (P8)

(P3) The isolated path property. For every computable tree $T \subseteq 2^{<\omega}$ with no terminal nodes and with isolated paths dense,

$$(\exists g \leq_T X) (\forall \sigma \in T) \left[ g_\sigma \in [T_\sigma] \land g_\sigma \text{ is isolated} \right].$$

(P4) The tree property. For every computable extendible tree $T \subseteq 2^{<\omega}$, and uniformly $\Delta^0_2$ sequence of subsets $\{S_i\}_{i \in \omega}$ dense in $T$, there exists $g \leq_T X$ for all $\sigma \in T$, $g_\sigma = \lambda y \left[ g(\sigma, y) \right]$ is a path extending $\sigma$ and hitting each $S_i$, i.e.,

$$(\exists g \leq_T X) (\forall \sigma \in T) \left[ \sigma \subset g_\sigma \land (\forall i) (\exists r \in S_i) [r \subset g_\sigma \in [T]] \right].$$
Topology

Property (P4) has a topological interpretation in the Cantor Space $2^\omega$. Recall in Cantor Space the basic open sets are

$$U_\sigma = \{ f : f \in 2^\omega \ & \sigma \subset f \}.$$ 

and open sets are

$$U_S = \bigcup \{ U_\sigma : \sigma \in S \}.$$ 

Hence, (P4) says that for every $\sigma \in T$, the path $g_\sigma \in [T]$ extends $\sigma$ and lies in every dense open set $U_{S_1}$. This says for the $\Delta_0^1$ family $\mathcal{G} = \{ S_i \}_{i \in \omega}$ that $X$ can compute a $\mathcal{G}$-generic path $g$. A special case is that $X$ computes a 1-generic set.
Omitting Types

(P5) *The omitting types property.* For any complete decidable theory $T$ and any uniformly $\Delta^0_2$ family of sets of formulas $\{\Gamma_j(\bar{x})\}_{j \in \omega}$, all nonprincipal with respect to $T$, there is an $X$-decidable model of $T$ omitting all $\Gamma_j(\bar{x})$.

**Def.** A set $S \subseteq \omega$ is $X$-monotonic if there is a function $g \leq_T X$ such that for every $x$, $g(x, y)$ is nondecreasing in $y$, with limit $\hat{g}(x) = \lim_y g(x, y)$, $\hat{g}(x) \geq x$, and $\hat{g}(x) \in S$.

(P6) *The monotonic property.* Every infinite $\Delta^0_2$ set $S$, is $X$-monotonic, i.e.,

$$\exists y \leq_T X \forall x \forall y \left[ x \leq g_x(y) \leq g_x(y+1) \land \lim_y g(x, y) \downarrow \in S \right].$$
Algebraic Properties

An equivalence structure is a structure of the form $\mathcal{A} = (A, E)$, where $E$ is an equivalence relation on $A$.

(P7) The equivalence structure property. For any $\Delta^0_2$ set $S \subseteq \omega - \{0\}$, there is an $X$-computable equivalence structure with one class of size $n$ for each $n \in S$, and no other classes.

(P8) The Abelian $p$-group property. For any infinite $\Delta^0_2$ set $S \subseteq \omega - \{0\}$, there is an $X$-computable reduced Abelian $p$-group $\mathcal{G}$, of length $\omega$, and with $u_n(\mathcal{G}) \leq 1$ for all $n$, such that $S(\mathcal{G}) = S$. 
Assume $X$ satisfies the escape property (P0),

(1) $(\forall h \leq_T 0') (\exists f \leq_T X) (\exists^\infty x) [ h(x) \leq f(x) ]$.

Let $T \subseteq 2^{<\omega}$ be a computable extendible tree with isolated paths dense. Define $g(\sigma, s) \leq_T X$

$$(\forall \sigma \in T)[ \sigma \subset g_\sigma \in [T]^{P} ].$$

Let $S$ be the set of atoms of $T$, i.e., nodes $\sigma$ with a unique extension $f \in [T_\sigma]$. Since $S$ is $\Pi^0_1$ and hence $\Delta^0_2$, there is a computable sequence $\{S_s\}_{s \in \omega}$ such that $S(x) = \lim_s S_s(x)$ for all $x$. Assume $\forall \tau \in T, \forall s, S_s$ contains some $\rho \supseteq \tau$.

For every $z \in T$ define the target,

$y_z = (\mu y)[z \subset y \& y \in S], \quad \text{and} \quad y^s_z = (\mu y)[z \subset y \& y \in S_s].
**Using the Escape Function**

**Def.** Define fn \( h \leq_T 0' \).

\[
h(n) = (\mu s) \ (\forall z)_{|z| \leq n} (\forall w \leq y_z^s) \]

\[
(\forall t \geq s)[ S_t(w) = S_s(w) = S(w)].
\]

(Note \( h \) total because \((\forall n) (\exists \inf z)[|z| \leq n] \).)

\((\forall z)(\forall s)\) the apparent target \( y_z^s \) stabilizes using \( S(x) = \lim_s S_s(x) \).)

By the escape property (P0) in (1),

\((\exists f \leq_T X)(\exists T\ \text{infinite}) (\forall t \in T)[ h(t) \leq f(t) ] \).

\( f \) is monotonic. Call \( T \) the set of true stages.
True Stages

Speed up to $X$-computable sequence $\hat{S}_x = S_{f(x)}$. Define $\hat{y}_z^s = g^{f(s)}_x$ $X$-computable in $z$ and $s$.

**Note.** Any apparent target $\hat{y}_z^t$ at a true stage $t \in T$ is the true target $y_z$, i.e.,

$$(\forall t \in T) (\forall z)_{|z| \leq t} (\forall v \geq t) [ \hat{y}_z^t = \hat{y}_z^v = y_z ].$$

For $s \leq |x|$ define $g(x, s) = x|s$. Fix $s \geq |x|$ and assume we are given $g(x, s)$ with $|g(x, s)| = s$.
Define $g(x, s+1) = \hat{g}^{g(x,s)}_x(s+1)$.

$$(\forall s > |x|)[x \subset g(x, s) \subset g(x, s+1) \land |g(x, s)| = s].$$
KEY POINT.

If $t \in T$ and $y = \hat{y}_{g(x,t)}^t$, then for every $s$ with $t < s < v = |y|$, we have $\hat{y}_{g(x,s)}^s = y$, because $y$ will be the most attractive target for $g(x, s)$ since no elements $w \leq y$ enter or leave $S$ after stage $t$.

Hence, if $t \in T$, then the sequence $\{ g(x, s) : t < s \leq v \}$ marches inexorably from $g(x, t)$ toward $y$ until hitting it at stage $v$, even though the intermediate stages $s$ with $t < s < v$, need not be in $T$. Hence, $g_x \in U_S$, and so $g_x$ is an isolated path.
Theories and $\Pi^0_1$-Classes

**Def.** $C \subseteq 2^\omega$ is a $\Pi^0_1$-class if there is a computable relation $R(x)$ such that

$$C = \{ f : (\forall x) R(f(x)) \}.$$

or equivalently a computable (not necessarily extendible) tree $T \subset 2^{\leq \omega}$ s.t.

$$C = [T].$$

**Thm.** If $T$ is an axiomatizable theory, then the class of complete extensions is $\Pi^0_1$. Put $\theta_{\alpha}$ on $T$ if $|\alpha| = s$ and

$$(\forall \beta \subseteq \alpha)[T_s \not\vdash \neg \theta_{\beta}],$$

i.e., if $\theta_{\alpha}$ seems consis with $T$ after $s$ steps.
Def. Peano Arithmetic (PA) is the first order theory of arithmetic with induction.

Cor. There is a computable tree $T_P \subseteq 2^{\langle \omega \rangle}$ such that

$$[T_P] \{ f : f \text{ a complete extension of PA } \}.$$

Thm (Jockusch-Soare). If $\mathcal{T} \subseteq 2^{<\omega}$ and $[\mathcal{T}] \neq \emptyset$, then

$$(\exists f \text{ low })[ f \in [\mathcal{T}] ].$$

Cor. $[T_P]$ contains a low complete extension of PA.
**PA Climbs Trees**

**Thm.** Let $[T]$ be a nonempty $\Pi^0_1$ class, and $g$ any complete extension of PA. Then

$$(\exists f \leq_T g) [ f \in [T] ].$$

**Proof.** Ask Rosser-type question about climbing a tree.

**Cor.** $(\exists$ low complete extension $g$ of PA)$

$$(\forall \Pi^0_1$ nonempty class $[T])

(\exists f \leq_T g) [ f \in [T] ].$$

**Cor.** The Low Basis Thm does not imply low prime models.
Def.
Def. Thm.
Def.
Def. Thm.