Computability of Vaughtian Models:

Lecture 1: Degrees of Prime Models

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Denumerable Models of Complete Theories $T$

Convention, Vaught [1961, §1–4].

• $T$ a countable complete theory.
• We consider only countable models $\mathcal{A} \models T$.

Vaughtian Models:

• **Prime** $\Rightarrow$ smallest model $\mathcal{A} \models T$ if $\exists$.
• **Atomic** $\Rightarrow$ Prime for $\mathcal{A}$ countable.
• **Saturated** $\Rightarrow$ largest $\mathcal{A} \models T$ if $\exists$.
• **Homogeneous** $\Rightarrow$ uniformly distributed.

Prime $\Rightarrow$ Homogeneous.

Saturated $\Rightarrow$ Homogeneous.

The period from the 1950’s through the 1960’s was extraordinarily fertile for both model theory and computability theory. A series of papers analyzed the concept of types in the Stone space $S(T)$ with ever more powerful results.


Morley further developed analysis of types for epochal theorem [1965].

Poizat [2000, p. 237] wrote,

“Morley, who by proving the first theorem on structure, . . . is the founder of stability theory, that is to say, of contemporary model theory.”
Computability Theory

Tutorial Notes, Chap. 2. Computability.

From new book:


(Also covered in Tutorial session.)
Building Models: Henkin

$T$ a (countable) consistent theory.

Thm. Godel (1930) $T$ has a model $A$.

Proof. Henkin (1949) Let $\mathcal{L} = \mathcal{L}(T)$.

1. Expand $\mathcal{L}_c = \mathcal{L} \cup \{ c_k \}_{k \in \omega}$, form $T_c$ by adding

*Henkin axioms* of the form

$$(\exists x)\theta(x) \rightarrow \theta(c)$$

2. *(Lindenbaum)* Take complete extn $T'_c$ of $T_c$.

3. Define the canonical model $A$ for $T'_c$.

Universe $A = \{ c^* : c \in \mathcal{L}_c \}$ where

$$c^* = \{ d : T'_c \vdash c = d \}.$$ 

Define function $F^A(c^*) = d^*$ iff $F(c) = d$ is a sentence of $T'_c$ likewise relations $R^A$.
Lindenbaum’s Lemma on a Tree

Def. (i) Let \( \{ \sigma_k \}_{k \in \omega} \) = all \( L_c \)-sentences. Let \( \sigma^1 \) denote \( \sigma \) and \( \sigma^0 \) denote \( \neg \sigma \). For \( \alpha \in 2^{<\omega} \) define

\[
\sigma_\alpha = \bigwedge_{i < |\alpha|} \sigma_i^{\alpha(i)}.
\]

(ii) Fix \( T_c \) an \( L_c \)-theory \( T_c \). Define the TREE

\[
\mathcal{T}_0(T_c) = \{ \sigma_\alpha : T_c \vdash \sigma_\alpha \},
\]

the tree of sentences provable in \( T_c \).

(iii) \( [\mathcal{T}_0(T_c)] = \) set of paths of \( \mathcal{T}_0(T_c) \).

Note. For any path \( f \in [\mathcal{T}_0(T_c)] \)

canonical model \( A_f \models T_c \).

(See diagram of tree \( \mathcal{T}_0(T_c) \).)
Decidable Models

Def. (i) If $T$ a theory and $\mathcal{A} \models T$ the elementary diagram $D^e(\mathcal{A})$ is the set of all $L_{\mathcal{A}}$-sentences true in $\mathcal{A}$.

(ii) Model $\mathcal{A}$ is decidable if $D^e(\mathcal{A})$ is.

Thm. If $T$ is a decidable theory, then $T$ has a decidable model.

Proof.
1. $T$ is decidable.
2. $T_c$ is decidable.
3. Tree $T_0(T_c)$ is decidable and extendible.
4. $\exists$ computable $f \in [T_0(T_c)]$.
5. Canonical model $\mathcal{A}_f$ is decidable.
This gives us a reliable way to build a variety of models but no way to guarantee that they are prime or saturated. To this we must control the types as we construct f.
Formulas and Types

\( T \) a complete decidable (CD) theory (now on).

\( F_n(T) = \{ \theta(x_0, x_2, \ldots x_{n-1}) \} \) fmlas in \( \mathcal{L}(T) \) with
\( n \) free variables and \( F(T) = \bigcup_n F_n(T) \).

\( B_n(T) = \text{Lindenbaum algebra} \) of equiv. classes
\( \theta(\bar{x})^* = \{ \psi(\bar{x}) : T \vdash (\forall \bar{x})[\theta(\bar{x}) = \psi(\bar{x})] \} \).

Let \( F(T) = \{ \theta_j(\bar{x}) \}_{j \in \omega} \). For \( \alpha \in 2^{< \omega} \) define
\[
\theta_\alpha(\bar{x}) = \bigwedge \{ \theta_i^{\alpha(i)}(\bar{x}) : i < |\alpha| \},
\]
for \( \theta^1 = \theta, \theta^0 = \neg \theta \).

**Tree.** \( T_n(T) = \{ \theta_\alpha(\bar{x}) : (\exists \bar{x})[\theta_\alpha(\bar{x})] \in F_n(T) \} \).
(Identify fmla \( \theta_\alpha \) and index \( \alpha \).)

\( S_n(T) \) is the set of \( n \)-types: \( S_n(T) = [T_n(T)] \).
Define \( S(T) = \bigcup_n S_n(T) \) all types.
This pix includes $F(t)$ and $S(T)$ and prin types (red) rank 0 and nonprin types (blue) either rank 1 or rank $\infty$. 
Atomic Trees

Def. $S_n(T) \subset 2^\omega$ has clopen sets,
$$U_\alpha = \{ f : f \in S_n(T) \& \ \alpha \subset p \}.$$ 

Def. Let $T$ be an extendible tree (every $\alpha \in T$ extends to $f \in [T]$).

(i) Nodes $\beta, \gamma \in T$ split node $\alpha$ if $\alpha \subset \beta, \alpha \subset \gamma,$ and $\beta \rhd \gamma.$

(ii) $\alpha \in T$ is an atom if $\alpha$ does not split, i.e.,
$$U_\alpha \cap [T] = \{ f \}, \text{ i.e., } \exists f \supset \alpha \in [T].$$
$\alpha$ is a generator of $f$; $\alpha$ isolates $f$.

(iii) Tree $T$ is atomic if
$$\forall \beta \in T \exists \alpha \supset \beta [\alpha \in T].$$
Def. For any model $\mathcal{A} \models T$ define

$$T(\mathcal{A}) = \{ p : p \in S(T) \ & \ \mathcal{A} \text{ realizes } p \}.$$ 

Def. A type $p \in S_n(T)$ is principal (isolated) if some atom in $T_n(T)$ isolates it. Define

$$S^P(T) = \{ p : p \text{ is a principal type of } S(T) \}.$$ 

Note. $S^P(T) \subseteq T(\mathcal{A})$ for every $\mathcal{A} \models T$.

Proof. Let $p(\overline{x})$ have generator $\theta(\overline{x})$. Then $(\exists \overline{x})\theta(\overline{x}) \in T$ since $T$ is complete.

Def. Model $\mathcal{A} \models T$ is atomic if every $a \in |\mathcal{A}|$ realizes a principal type, i.e.,

$$T(\mathcal{A}) = S^P(T).$$
Prime Models and Atomic Theories

Def. A theory $T$ is **atomic** if $T_n(T)$ is atomic for every $n$.

Def. $A \models T$ is **prime** if $A \preceq B$ for every $B \models T$.

Thm. (Vaught) $A$ is prime iff countable and atomic.

Thm. (Vaught) A theory $T$ has a prime model iff $T$ is atomic.

Def. $A$ is **homogeneous** iff for all $\bar{a}$, and $\bar{b}$,

$$(A, \bar{a}) \equiv (A, \bar{b}) \implies (\exists G \in \text{Aut}(A))[G(\bar{a}) = \bar{b}].$$

Thm. (Vaught) If $A$ is prime $A$ is homogeneous.
Example: Atomic Theory

Def. DLO
Theory of dense linear orderings w/o endpoints.

Example $A = (A, <), A =$ rationals.

Take theory $T = Th(A, <, A)$.

$|S_1(T)| = 2^\aleph_0$ but $T$ is atomic.

Isolated (principal) type: $x = q$

Nonisolated type: $p(x)$ for $x = \sqrt{2}$. We write:

$p(x) = \{x < q : q^2 > 2\} \cup \{x > q : q^2 < 2\}$.

(See diagram of atomic tree.)
Diagram 1: Fiding Atoms

$\mathcal{T}$: A computable extendible atomic tree.

Search. $(\forall \sigma \in \mathcal{T})(\exists \text{ atom } \tau \supseteq \sigma)[ \tau \in \mathcal{T}].$
Diagram 1: Uniform Matrix

$T$: A Complete Decidable (CD) theory, $A \models T$.

From $D^e(A)$ we get uniform listing of all types $T(A)$ namely \{\(p(\pi) : \pi \in A\}\).
Decidable Prime Models

Let $T$ be complete atomic decidable (CAD).

**Thm. (Millar)** There is a CAD theory $T$ with no decidable prime model.

**Def.** A 0-basis for a countable class $\{A_y\}_{y \in \omega}$ is a function $g \leq 0$ s.t. $g(x, y) = A_y(x)$.
(Uniformly computable matrix $g$.)

**Thm. (Goncharov-Nurtazin, Harrington)** If $T$ is a CAD theory TFAE:

(i) $T$ has a decidable prime model $A$.

(ii) $S^P(T)$ has a 0-basis.

(iii) $(\exists g \leq_T 0)(\forall \theta_\alpha \in T_n(T))[\theta_\alpha \subset g_\alpha \in S^P_n(T)]$, where $g_\alpha = \lambda y [g(\alpha, y)]$ is path in $[T_n(T)]$. 

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Proof sketch. (i) ⇒ (ii). Let $\mathcal{A}$ be decidable, 

$$\mathcal{T}(\mathcal{A}) = \{ p(\bar{\pi}) : \bar{\pi} \in \mathcal{A} \}.$$  

(ii) ⇒ (iii). Given $\{p_i\}_{i \in \omega} = \mathcal{T}(\mathcal{A})$ and $\theta_\alpha \in \mathcal{T}_n(T)$ find first $n$-type $p_i$ with $\theta_\alpha \in p_i$.

(iii) ⇒ (ii). As $\theta_\alpha$ ranges through the $\mathcal{T}_n(T)$ path $g_\alpha$ ranges through $S^P_n(T)$ uniformly in $n$.

(ii) ⇒ (i).** This is the main import of the theorem, a priority argument.

Remark. (ii) ⇒ (i) not obvious. Choose a 1-type $p_1$, put $p(c_1)$ into $D^\alpha(\mathcal{A})$.

For $(c_1, c_2)$ we need 2-type $p_2$ consistent with $p_1$, then put $p_2(c_1, c_2)$ into $D^\alpha(\mathcal{A})$.

But consistency of $p_1$ and $p_2$ is a $\Pi_1$ property, decidable in $\mathcal{O}'$. 

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Proof of (ii) $\implies$ (i)

Consider a limit $p_2^s$ for the correct 2-type. As each one fails, move to the next one which is $s$-consistent with $p_1^s(c_1)$.

Eventually come to atom $\alpha \in p_1$ and thereafter no injury.
**Undecidable Prime Models**

**Def.** $0' = K = \{ e : e \in W_e \}$, halting problem.

**Thm.** If $T$ is a CAD theory $T$ has a prime model decidable in $0'$.

**Prf.** $0'$ can decide (ii) or (iii) of previous criterion relative to $0'$, or apply original Vaught proof with $0'$ as oracle to detect atoms.

**Def.** $dgSp(A) = \{ \deg(D^c(B)) : B \cong A \}$. *degree spectrum of* $A$.

**Thm (Knight).** If $A$ is a countable structure in a relational language, then either $dgSp(A)$ is a singleton, or $dgSp(A)$ is closed upwards.
Consider pix of degrees below $K$.
$K$ is red (we can find prime model below $K$)
0 is blue (we cannot.)
Now consider entire class of Delta-2 degrees.
Turing Reducibility

Oracle Machines

Def. Thm.
Def. A is limit computable if there is a uniformly computable sequence \( \{A_s\}_{s \in \omega} \) s.t. 
\( A = \lim_s A_s \). (See diagram).

Limit Lemma. TFAE:

(i) \( f \) is limit computable.
(ii) \( A \leq_T \emptyset' \).
(iii) \( A \in \Delta_2 \).
Degrees ≤0′

Def.
Def. Thm.
Forcing and 1-Generic Sets

Def.  Given c.e. set $V_e \subseteq 2^{<\omega}$. We say $f \in 2^{\omega}$ forces $V_e$ if we satisfy the forcing requirement,

$$F_e : (\exists \sigma \subset f)[ \sigma \in V_e \lor (\forall \rho \supset \sigma)[ \rho \notin V_e ]] .$$

We say that $\sigma$ forces $F_e$ and any $f \supset \sigma$ forces $F_e$.

(ii) $f$ is 1-generic if $f$ forces $W_e$, all $e \in \omega$.

Thm.  A set $A$ is 1-generic iff $A$ satisfies for all $e$ following requirement $J_e$ which is called forcing the jump

$$J_e : (\exists \sigma \subset A)[ \Phi^e_\sigma(e) \downarrow \lor (\forall \tau \supseteq \sigma)[ \Phi^e_\tau(e) \uparrow ]] .$$

(We decide whether $e \in A'$.)
Prime Model Low Basis Theorem

**Thm. (Csima)** Every complete decidable atomic (CAD) theory $T$ has a low prime model (i.e., $D^c(A)$ is low).

Build $f = D^c(A)$ by $0'$-construction which forces jump on $f = \cup f_s$.

**Stage $s+1$.** Given $f_s$.

**Step 1.** Find an extension $f' \supset f_s$, $f' \in T_0(T)$, which forces the jump.

**Step 2.** Find $f'' \supset f'$, $f'' \in T_0(T)$, s.t. it forces $\langle c_0, c_1, \ldots, c_s \rangle$ to satisfy an atom of $T_s(T)$. Let $f_{s+1} = f''$.

**Refutes: Conjecture [Clote].** There is a CAD theory $T$ s.t. $D^c(A) \geq T 0'$ for every prime $A \models T$.

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**Thm (Millar).** There is a CAD theory $T$ with no decidable prime model.

**Proof.** By Harrington it suffices to construct $T$ such that there is no uniformly computable listing of $S^P(T)$, i.e., no computable matrix $g(x, y)$ such that

$$S^P(T) = \{g_y\}_{y \in \omega}$$

where row $g_y = \lambda x \lfloor g(x, y) \rceil$.

**Game.** This is a game between

- $T$-Player. Plays a CAD theory $T$.
- $A$-Player. Plays a prime model $A \models T$.

**Win.** The $A$-player wins if $A$ has the desired degree (computable, $0'$, low, etc.) and the $T$-player wins otherwise.