# Computability of Vaughtian Models:

## Lecture 1: Degrees of Prime Models

**by**

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## Vaught 1961

*Denumerable Models of Complete Theories T*

**Convention, Vaught [1961, §1–4].**

- $T$ a countable complete theory.
- We consider only *countable* models $\mathcal{A} \models T$.

**Vaughtian Models:**

- **Prime** = smallest model $\mathcal{A} \models T$ if $\exists$.
- **Atomic** = Prime for $\mathcal{A}$ countable.
- **Saturated** = largest $\mathcal{A} \models T$ if $\exists$.
- **Homogeneous** = uniformly distributed.

Prime $\implies$ Homogeneous.

Saturated $\implies$ Homogeneous.

See Soare Tutorial Chap. 1 *Model Theory*, pp. 3–21, and tutorial session Mon. evening.

The period from the 1950's through the 1960's was extraordinarily fertile for both model theory and computability theory. A series of papers analyzed the concept of types in the Stone space $S(T)$ with ever more powerful results.

MODEL THEORY: Vaught [1958] [1961]*, [1963]. D. Marker [2002, p. 172] described [1961] as "one of the most elegant papers in model theory,"

Morley further developed analysis of types for epochal theorem [1965].

Poizat [2000, p. 237] wrote,

"Morley, who by proving the first theorem on structure, . . . is the founder of stability theory, that is to say, of contemporary model theory."

## Computability Theory

Tutorial Notes, Chap. 2. Computability.

From new book:

Soare, *Computability Theory and Applications*, Springer-Verlag, Heidelberg, to appear.

(Also covered in Tutorial session.)

## Building Models: Henkin

**T** a (countable) consistent theory.

**Thm. Godel (1930)** $T$ has a model $\mathcal{A}$.

**Proof. Henkin (1949)** Let $\mathcal{L} = \mathcal{L}(T)$.

1. Expand $\mathcal{L}_c = \mathcal{L} \cup \{ c_k \}_{k \in \omega}$, form $T_c$ by adding *Henkin axioms* of the form

$$(\exists x)\theta(x) \longrightarrow \theta(c)$$

2. **(Lindenbaum)** Take complete extn $T'_c$ of $T_c$.

3. Define the canonical model $\mathcal{A}$ for $T'_c$.
Universe $A = \{ c^* : c \in \mathcal{L}_c \}$ where

$$c^* = \{ d \ : \ T'_c \vdash c = d \}.$$

Define function $F^{\mathcal{A}}(c^*) = d^*$ iff $F(c) = d$ is a sentence of $T'_c$ likewise relations $R^{\mathcal{A}}$.

## Lindenbaum's Lemma on a Tree

**Def.** (i) Let $\{\sigma_k\}_{k\in\omega}$ = all $\mathcal{L}_c$-sentences. Let $\sigma^1$ denote $\sigma$ and $\sigma^0$ denote $\neg\sigma$. For $\alpha \in 2^{<\omega}$ define

$$\sigma_\alpha \;=\; \wedge_{i<|\alpha|}\, \sigma_i^{\alpha(i)}.$$

(ii) Fix $T_c$ an $\mathcal{L}_c$-theory $T_c$. Define the **TREE**

$$\mathcal{T}_0(T_c) \;=\; \{\sigma_\alpha \;:\; T_c \vdash \sigma_\alpha\},$$

the tree of sentences provable in $T_c$.

(iii) $[\mathcal{T}_0(T_c)]$ = set of paths of $\mathcal{T}_0(T_c)$.

**Note.** For any path $f \in [\mathcal{T}_0(T_c)]$

canonical model $\mathcal{A}_f \models T_c$.

(See diagram of tree $\mathcal{T}_0(T_c)$.)

## Decidable Models

**Def.** (i) If $T$ a theory and $\mathcal{A} \models T$ the *elementary diagram* $D^e(\mathcal{A})$ is the set of all $\mathcal{L}_A$-sentences true in $\mathcal{A}$.

(ii) Model $\mathcal{A}$ is *decidable* if $D^e(\mathcal{A})$ is.

**Thm.** If $T$ is a decidable theory, then $T$ has a decidable model.

**Proof.**

1. $T$ is decidable.
2. $T_c$ is decidable.
3. Tree $\mathcal{T}_0(T_c)$ is decidable and extendible.
4. $\exists$ computable $f \in [\mathcal{T}_0(T_c)]$.
5. Canonical model $\mathcal{A}_f$ is decidable.

This gives us a reliable way to build a variety of models but no way to guarantee that they are prime or saturated. To this we must control the types as we construct f.

## Formulas and Types

**T** a complete *decidable* (CD) theory (now on).

$\mathbf{F_n(T)} = \{\, \theta(x_0, x_2, \ldots x_{n-1})\,\}$ fmlas in $\mathcal{L}(T)$ with $n$ free variables and $F(T) = \cup_n F_n(T)$.

$\mathbf{B_n(T)}$ = *Lindenbaum algebra* of equiv. classes $\theta(\overline{x})^* = \{\, \psi(\overline{x}) \,:\, T \vdash (\forall \overline{x})\,[\,\theta(\overline{x}) = \psi(\overline{x})\,]\,\}$.

Let $F(T) = \{\theta_j(\overline{x})\}_{j\in\omega}$. For $\alpha \in 2^{<\omega}$ define

$$\theta_\alpha(\overline{x}) \quad = \quad \bigwedge\!\!\!\bigwedge \{\, \theta_i^{\alpha(i)}(\overline{x}) \,:\, i < |\alpha|\,\},$$

for $\theta^1 = \theta$, $\theta^0 = \neg\,\theta$.

**Tree.** $\mathcal{T}_n(T) = \{\, \theta_\alpha(\overline{x}) \,:\, (\exists \overline{x})[\,\theta_\alpha(\overline{x})\,] \in F_n(T)\,\}$. (Identify fmla $\theta_\alpha$ and index $\alpha$.)

$\mathbf{S_n(T)}$ is the set of *n-types:* $S_n(T) = [\,\mathcal{T}_n(T)\,]$.

Define $S(T) = \cup_n S_n(T)$ all types.

This pix includes F(t) and S(T) and prin types (red) rank 0 and nonprin types (blue) either rank 1 or rank $\infty$.

## Atomic Trees

**Def.** $S_n(T) \subset 2^\omega$ has clopen sets,

$$\mathcal{U}_\alpha = \{\, f \,:\, f \in S_n(T) \;\;\&\;\; \alpha \subset p \,\}.$$

**Def.** Let $\mathcal{T}$ be an extendible tree (every $\alpha \in \mathcal{T}$ extends to $f \in [\mathcal{T}]$).

(i) Nodes $\beta$, $\gamma \in \mathcal{T}$ *split* node $\alpha$ if $\alpha \subset \beta$, $\alpha \subset \gamma$, and $\beta \,|\, \gamma$.

(ii) $\alpha \in \mathcal{T}$ is an *atom* if $\alpha$ does not split, *i.e.*,

$$\mathcal{U}_\alpha \;\cap\; [\,\mathcal{T}\,] \quad = \quad \{\, f \,\}, \; \textit{i.e.},$$

$$(\,\exists\,!\; f \supset \alpha\,)\,[\; f \in [\,\mathcal{T}\,]\;].$$

$\alpha$ is a *generator* of $f$; $\alpha$ *isolates* $f$.

(iii) Tree $\mathcal{T}$ is *atomic* if

$$(\forall \beta \in \mathcal{T})(\exists \alpha \supset \beta)[\alpha \in \mathcal{T}].$$

## Principal Types and Atomic Models

**Def.** For any model $\mathcal{A} \vDash T$ define

$$\mathbb{T}(\mathcal{A}) = \{p : p \in S(T) \ \ \& \ \ \mathcal{A} \text{ realizes } p\}.$$

**Def.** A type $p \in S_n(T)$ is *principal (isolated)* if some atom in $\mathcal{T}_n(T)$ isolates it. Define

$$S^P(T) = \{ \, p \ : \ p \ \text{ is a principal type of } S(T) \, \}.$$

**Note.** $S^P(T) \subseteq \mathbb{T}(\mathcal{A})$ for every $\mathcal{A} \vDash T$.

**Proof.** Let $p(\overline{x})$ have generator $\theta(\overline{x})$. Then $(\exists\overline{x})\theta(\overline{x}) \in T$ since $T$ is complete.

**Def.** Model $\mathcal{A} \vDash T$ is *atomic* if every $a \in |\mathcal{A}|$ realizes a principal type, *i.e.*,

$$\mathbb{T}(\mathcal{A}) \quad = \quad S^P(T).$$

## Prime Models and Atomic Theories

**Def.** A theory $T$ is *atomic* if $\mathcal{T}_n(T)$ is atomic for every $n$.

**Def.** $\mathcal{A} \models T$ is *prime* if $\mathcal{A} \preceq \mathcal{B}$ for every $\mathcal{B} \models T$.

**Thm. (Vaught)** $\mathcal{A}$ is prime iff countable and atomic.

**Thm. (Vaught)** A theory $T$ has a prime model iff $T$ is atomic.

**Def.** $\mathcal{A}$ is *homogeneous* iff for all $\overline{a}$, and $\overline{b}$,

$$(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b}) \implies (\exists G \in \mathrm{Aut}(\mathcal{A}))[G(\overline{a}) = \overline{b}].$$

**Thm. (Vaught)** If $\mathcal{A}$ is prime $\mathcal{A}$ is homogeneous.

## Example: Atomic Theory

**Def. DLO**
Theory of dense linear orderings w/o endpoints.

**Example** $\mathcal{A} = (A, <)$, $A =$ rationals.

Take **theory** $T = Th(\mathcal{A}, <, A)$.

$|S_1(T)| = 2^{\aleph_0}$ but $T$ is atomic.

**Isolated (principal) type**: $x = q$

**Nonisolated type**: $p(x)$ for $x = \sqrt{2}$. We write:
$p(x) = \{x < q : q^2 > 2\} \cup \{x > q : q^2 < 2\}$.

(See diagram of atomic tree.)

## Diagram 1: Fiding Atoms

$\mathcal{T}$: A computable extendible atomic tree.

**Search.** $(\forall\sigma \in \mathcal{T})(\exists \text{ atom } \tau \supseteq \sigma)[\ \tau \in \mathcal{T}\ ]$.

## Diagram 1: Uniform Matrix

$\mathcal{T}$: A Complete Decidable (CD) theory, $\mathcal{A} \models T$.

From $D^e(\mathcal{A})$ we get uniform listing of all types $\mathbb{T}(\mathcal{A})$ namely $\{p(\overline{a}) : \overline{a} \in \mathcal{A}\}$.

## Decidable Prime Models

Let $T$ be *complete atomic decidable* (CAD).

**Thm. (Millar)** There is a CAD theory $T$ with no decidable prime model.

**Def.** A $\mathbf{0}$-*basis* for a countable class $\{A_y\}_{y\in\omega}$ is a function $g \leq \mathbf{0}$ s.t. $g(x,y) = A_y(x)$.
(Uniformly computable matrix $g$.)

**Thm. (Goncharov-Nurtazin, Harrington)**
If $T$ is a CAD theory TFAE:

(i) $T$ has a decidable prime model $\mathcal{A}$.

(ii) $S^P(T)$ has a $\mathbf{0}$-basis.

(iii) $(\exists g \leq_{\mathrm{T}} \mathbf{0})(\forall \theta_\alpha \in \mathcal{T}_n(T))[\theta_\alpha \subset g_\alpha \in S^P_n(T)]$,
where $g_\alpha = \lambda y\, [g(\alpha, y)]$ is path in $[\mathcal{T}_n(T)]$.

## Proving the Criterion

**Proof sketch**. (i) $\Longrightarrow$ (ii). Let $\mathcal{A}$ be decidable,

$$\mathbb{T}(\mathcal{A}) \; = \; \{ \; p(\overline{a}) \; : \; \overline{a} \in \mathcal{A} \; \}.$$

(ii) $\Longrightarrow$ (iii). Given $\{p_i\}_{i \in \omega} = \mathbb{T}(\mathcal{A})$ and $\theta_\alpha \in \mathcal{T}_n(T)$ find first $n$-type $p_i$ with $\theta_\alpha \in p_i$.

(iii) $\Longrightarrow$ (ii). As $\theta_\alpha$ ranges through the $\mathcal{T}_n(T)$ path $g_\alpha$ ranges through $S_n^P(T)$ uniformly in $n$.

(ii) $\Longrightarrow$ (i).** This is the *main import* of the theorem, a priority argument.

**Remark.** (ii) $\Longrightarrow$ (i) not obvious. Choose a 1-type $p_1$, put $p(c_1)$ into $D^e(\mathcal{A})$.

For $(c_1, c_2)$ we need 2-type $p_2$ consistent with $p_1$, then put $p_2(c_1, c_2)$ into $D^e(\mathcal{A})$.

But consistency of $p_1$ and $p_2$ is a $\Pi_1$ property, decidable in $\mathbf{0}'$.

## Proof of (ii) $\implies$ (i)

Consider a limit $p_2^s$ for the correct 2-type. As each one fails, move to the next one which is $s$-consistent with $p_1^s(c_1)$.

Eventually come to atom $\alpha \in p_1$ and thereafter no injury.

## Undecidable Prime Models

**Def.** $\mathbf{0}' = K = \{e : e \in W_e\}$, halting problem.

**Thm.** If $T$ is a CAD theory $T$ has a prime model decidable in $\mathbf{0}'$.

**Prf.** $\mathbf{0}'$ can decide (ii) or (iii) of previous criterion relative to $\mathbf{0}'$, or apply original Vaught proof with $\mathbf{0}'$ as oracle to detect atoms.

**Def.** $dgSp(\mathcal{A}) = \{ deg(D^e(\mathcal{B})) : \mathcal{B} \cong \mathcal{A} \}$. *degree spectrum of* $\mathcal{A}$.

**Thm (Knight).** If $\mathcal{A}$ is a countable structure in a relational language, then either $dgSp(\mathcal{A})$ is a singleton, or $dgSp(\mathcal{A})$ is closed upwards.

Consider pix of degrees below K.
K is red (we can find prime model below K)
0 is blue (we cannot.)
Now consider entire class of Delta-2 degrees.

## Turing Reducibility

Oracle Machines

**Def.** **Thm.**

## Limit Computable Sets

**Def.** $A$ is *limit computable* if there is a uniformly computable sequence $\{A_s\}_{s\in\omega}$ s.t. $A = \lim_s A_s$. (See diagram).

**Limit Lemma.** TFAE:

(i) $f$ is limit computable.

(ii) $A \leq_{\mathrm{T}} \emptyset'$.

(iii) $A \in \Delta_2$.

## Degrees $\leq$0′

**Def.**

**Def.** **Thm.**

## Forcing and 1-Generic Sets

**Def.** Given c.e. set $V_e \subseteq 2^{<\omega}$. We say $f \in 2^\omega$ *forces* $V_e$ if we satisfy the *forcing* requirement,

$$F_e : \ (\exists \sigma \subset f) [\, \sigma \in V_e \ \vee \ (\forall \rho \supset \sigma) [\, \rho \notin V_e \,] \,].$$

We say that $\sigma$ *forces* $F_e$ and any $f \supset \sigma$ *forces* $F_e$.

(ii) $f$ is *1-generic* if $f$ forces $W_e$, all $e \in \omega$.

**Thm.** A set $A$ is 1-generic iff $A$ satisfies for all $e$ following requirement $J_e$ which is called *forcing the jump*

$$J_e : \ (\exists \sigma \subset A) [\, \Phi_e^\sigma(e)\!\downarrow \ \vee (\forall \tau \supseteq \sigma) [\, \Phi_e^\tau(e)\!\uparrow \,] \,].$$

(We decide whether $e \in A'$.)

## Prime Model Low Basis Theorem

**Thm. (Csima)** Every complete decidable atomic (CAD) theory $T$ has a low prime model (*i.e.*, $D^e(\mathcal{A})$ is low).

Build $f = D^e(\mathcal{A})$ by $\mathbf{0}'$- construction which forces jump on $f = \cup f_s$.

*Stage s+1.* Given $f_s$.

*Step 1.* Find an extension $f' \supset f_s$, $f' \in \mathcal{T}_0(T)$, which forces the jump.

*Step 2.* Find $f'' \supset f'$, $f'' \in \mathcal{T}_0(T)$, s.t. it forces $\langle c_0, c_1, \dots c_s \rangle$ to satisfy an atom of $\mathcal{T}_s(T)$. Let $f_{s+1} = f''$.

**Refutes: Conjecture [Clote]**. There is a CAD theory $T$ s.t. $D^e(\mathcal{A}) \geq_{\mathrm{T}} 0'$ for every prime $\mathcal{A} \models T$.

## No Decidable Prime Model

**Thm (Millar).** There is a CAD theory $T$ with no decidable prime model.

**Proof.** By Harrington it suffices to construct $T$ such that there is no uniformly computable listing of $S^P(T)$, *i.e.*, no computable matrix $g(x, y)$ such that

$$S^P(T) \;=\; \{g_y\}_{y \in \omega}$$

where row $g_y = \lambda x\,[\,g(x, y)\,]$.

**Game.** This is a game between

*$T$-Player.* Plays a CAD theory $T$.

*$\mathcal{A}$-Player.* Plays a prime model $\mathcal{A} \models T$.

*Win.* The $\mathcal{A}$-player *wins* if $\mathcal{A}$ has the desired degree (computable, $\mathbf{0}'$, low, etc.) and the $T$-player wins otherwise.