THE UNIVERSITY OF CHICAGO

ON THE ABELIAN SANDPILE MODEL

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
EVELIN CHRISTIANA TOUMPAKARI

CHICAGO, ILLINOIS
JUNE 2005

To the encounter of sandpiles with the turquoise Mediterranean.

ABSTRACT

The Abelian Sandpile Model is a diffusion process on graphs, studied, under various names, in statistical physics, theoretical computer science, and algebraic graph theory. The model takes a rooted directed multigraph $\mathcal{X}$, the *ambient space*, in which the root is accessible from every vertex, and associates with it a commutative monoid $\mathcal{M}$, a commutative semigroup $\mathcal{S}$, and an abelian group $\mathcal{G}$ as follows. For vertices i, j, let a_{ij} denote the number of $i \to j$ edges and let $\deg(i)$ denote the out-degree of i. Let V be the set of ordinary (non-root) vertices. With each $i \in V$ associate a symbol x_i and consider the relations $\deg(i)x_i = \sum_{j \in V} a_{ij}x_j$. Let $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{G}$ be the commutative monoid, semigroup and group, respectively, generated by $\{x_i : i \in V\}$ subject to these defining relations. $\mathcal{M}$ is the *sandpile monoid*, $\mathcal{S}$ is the *sandpile semigroup*, and $\mathcal{G}$ is the *sandpile group* associated with $\mathcal{X}$. We write the operation additively, so 0 is the identity of $\mathcal{M}$. We have $\mathcal{M} = \mathcal{S} \cup \{0\}$; we show that $\mathcal{G}$ is the unique minimal ideal of $\mathcal{M}$.

The main results of the thesis cover two areas: (1) a general study of the structure of the sandpile monoid and (2) detailed analysis of the structure of the sandpile group for a special class of graphs.

Our *first main goal* is to establish connections between the algebraic structure of $\mathcal{M}$, $\mathcal{S}$, $\mathcal{G}$, and the combinatorial structure of the underlying ambient space $\mathcal{X}$. $\mathcal{M}$ turns out to be a distributive lattice of semigroups each of which has a unique idempotent. The distributive lattice in question is the lattice $\mathcal{L}$ of idempotents of $\mathcal{M}$; $\mathcal{L}$ turns out to be isomorphic to the dual of the lattice of ideals of the poset of normal strong components of $\mathcal{X}$. The $\mathcal{M} \to \mathcal{L}$ epimorphism defines the *smallest* semilattice congruence of $\mathcal{M}$; therefore $\mathcal{L}$ is the *universal semilattice* of $\mathcal{M}$.

We characterize the directed graphs $\mathcal{X}$ for which $\mathcal{S}$ has a unique idempotent; this includes the important case when the digraph induced on the ordinary vertices is strongly connected. If the idempotent in $\mathcal{S}$ is unique then the Rees quotient $\mathcal{S}/\mathcal{G}$

(obtained by contracting $\mathcal{G}$ to a zero element) is nilpotent. Let, in this case, k denote the nilpotence class of $\mathcal{S}/\mathcal{G}$. We prove that there exist functions ψ_1 and ψ_2 such that $|\mathcal{S}/\mathcal{G}| \le \psi_1(k)$ and $\mathcal{G}$ contains a cyclic subgroup of index $\le \psi_2(k)$. This result is a corollary to our asymptotic characterization of the ambient spaces with bounded k: every sufficiently large directed multigraph with this property can be described as a "circular tollway system of bounded effective volume."

The sandpile group has been the subject of extensive study for various special classes of graphs, including the square lattice and the n-dimensional cube. Our *second main goal* is a detailed analysis of the sandpile group $G(d,h)$ for the case where $\mathcal{X}$ is the complete d-regular tree of depth h with a sink vertex attached $(d-1)$-fold to each leaf, so that all tree vertices have degree d in $\mathcal{X}$. We compute the rank, the exponent, the order, and other structural parameters of the abelian group $G(d,h)$. We find a cyclic Hall-subgroup of order $(d-1)^h$. We prove that the rank of $G(d,h)$ is $(d-1)^h$ and that $G(d,h)$ contains a subgroup isomorphic to $\mathbb{Z}_d^{(d-1)^h}$; therefore, for all primes p dividing d, the rank of the Sylow p-subgroup is maximal (equal to the rank of the entire group). We find that the base-$(d-1)$ logarithm of the exponent and of the order are asymptotically $3h^2/\pi^2$ and $c_d(d-1)^h$, respectively. We conjecture an explicit formula for the ranks of all Sylow subgroups.

ACKNOWLEDGEMENTS

I wish to thank my advisor Steven Lalley for introducing me to the Abelian Sandpile Model and for many discussions over the past several years. I am indebted to my coadvisor László Babai for taking great interest in my subject and for tireless mentoring through all phases of this work. I would like to thank Robert Fefferman for his continuous support during my graduate career. I wish to thank Dmitriy Cherkashin and Charilaos Skiadas for helpful discussions at the early stages of this research.

I am grateful to my parents Giorgos and Toula and to my brother Kostas, for my growing up in a wonderful family, for all their love and for always encouraging me to follow my way; and to my aunt Nasi, for always being just a flight of stairs away. I thank my awesome friend Olga Sergiyenko for lots of tea and workouts; my friends in Greece, Konstantina Plakia, Giannis Roumeliotis, Khara Falara, Nancy Malama, Katerina Dimitriadi, for big laughs and for sharing their youth with me; and Dima for the bright colors.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1
INTRODUCTION

1.1 The Abelian Sandpile Model

The Abelian Sandpile Model is a diffusion process on graphs, studied, under various names, in statistical physics, theoretical computer science, and algebraic graph theory[1]. The model takes a finite directed multigraph[2] ("digraph") $\mathcal{X}^*$ with a special vertex called the *sink* as its *ambient space*, and associates with it a finite commutative monoid[3] $\mathcal{M}$, a finite commutative semigroup $\mathcal{S}$, and a finite abelian group $\mathcal{G}$, called the *sandpile monoid*, the *sandpile semigroup* and the *sandpile group* of $\mathcal{X}^*$, respectively. We assume that the sink is accessible from every vertex and has out-degree zero. Vertices other than the sink will be called *ordinary.* A *state* of the game is an assignment of an integer $h_i \geq 0$ to each ordinary vertex i. The integer h_i may be thought of as the number of sandgrains (the *height* of the sandpile) at *site* i. A state is *stable* if for all ordinary vertices i, $0 \leq h_i < \deg_*(i)$, where $\deg_*$ denotes the out-degree. (We reserve "deg" to denote the degree relative to the set of ordinary vertices.) If $h_i \geq \deg_*(i)$ for an ordinary vertex i, the "pile" at i may be "toppled," sending one grain through each edge leaving i. So h_i is reduced by $\deg_*(i)$, and for each ordinary vertex j, the height h_j increases by the number of edges joining i to j. The sink "collects" the grains "falling off" the ordinary vertices and never topples. Starting with any state and toppling unstable ordinary vertices in succession,

---

1. The Abelian Sandpile Model is identical with the "dollar-game" in Biggs' terminology [6], and is a variant of the "chip-firing game" studied in computer science [8, 34].

2. A directed multigraph consists of a set V of vertices, a set E of edges, and an incidence function $i : E \to V^2$, where V^2 is the set of ordered pairs of vertices. For an edge $e \in E$, we write $i(e) = (h(e), t(e))$ and we call $h(e)$ the *head* of e and $t(e)$ the *tail* of e.

3. A *monoid* is a semigroup with identity.

we arrive at a stable state in a finite number of steps, since the sink is accessible from every vertex.

By a Jordan-Hölder argument (the "Diamond Lemma," cf. [31]), the order in which the topplings occur does not matter [8, 12]; given an initial state $\mathbf{h}$, every stabilizing sequence leads to the same stable state $\sigma(\mathbf{h})$; hence the term "abelian." The stabilization process is referred to by physicists as an "avalanche."

The *sandpile monoid* is defined as the set of stable states under the operation of pointwise addition and stabilization. We denote this operation by $\oplus$. So, for stable states $\mathbf{h}_1$ and $\mathbf{h}_2$ we set $\mathbf{h}_1 \oplus \mathbf{h}_2 := \sigma(\mathbf{h}_1 + \mathbf{h}_2)$. The all-zero state 0 is the identity in $\mathcal{M}$. The subsemigroup of $\mathcal{M}$ generated by the non-zero states is the *sandpile semigroup* $\mathcal{S}$. Clearly, $\mathcal{M} = \mathcal{S} \cup \{0\}$.

We say that the stable state $\mathbf{h}_1$ is *accessible* from the stable state $\mathbf{h}_2$ if $(\exists \mathbf{h} \in \mathcal{M})(\mathbf{h}_1 = \mathbf{h}_2 \oplus \mathbf{h})$. We say that a state $\mathbf{h}_1$ is accessible from a state $\mathbf{h}_2$ if $\sigma(\mathbf{h}_1)$ is accessible from $\sigma(\mathbf{h}_2)$. A stable state is called *recurrent* (or "critical") if it is accessible from every state. A stable state that is not recurrent is called *transient*. The set $\mathcal{G}$ of recurrent states turns out to be the unique minimal ideal of $\mathcal{M}$ and as such it is an abelian group (see Fact 2.3.4). The abelian group $\mathcal{G}$, introduced by Dhar [12] in 1990 (cf. [10]), is the *sandpile group*, a remarkable algebraic invariant of the directed graph $\mathcal{X}^*$.

1.2 The sandpile group and the Laplacian

The sandpile group is closely related to the *Laplacian* of the digraph $\mathcal{X}^*$. The Laplacian of a directed multigraph on n vertices is the $n \times n$ matrix $L = (L_{ij})$ where L_{ii} is the out-degree of the vertex i, and for $i \neq j$, $-L_{ij}$ is the number of edges from i to j. Note that the sum of every row of the Laplacian is zero. The *reduced* Laplacian is the matrix obtained from the Laplacian by deleting the row and the column corresponding to the sink. Let Λ be the lattice spanned by the rows of the reduced Laplacian. The sandpile group $\mathcal{G}$ is isomorphic to $\mathbb{Z}^{n-1}/\Lambda$. The order of $\mathcal{G}$ is the determinant of the reduced Laplacian, which, according to the digraph version of Kirchhoff's celebrated "Matrix-Tree Theorem," equals the number of directed spanning trees of our digraph, directed toward the root. We note that if $\mathcal{X}^*$ is undirected then the sandpile group does not depend on which vertex of $\mathcal{X}^*$ is selected to be the sink [9].

1.3 Related work

Most of the existing work on the Abelian Sandpile Model is regarding undirected ambient spaces.

The ambient spaces of greatest interest to physicists for sandpiles are the ones obtained from the square grid as follows: $\mathcal{X}^*_{n,m}$ is obtained by taking the $n \times m$ grid, adding an extra vertex (the sink) and attaching the sink to each vertex on the perimeter of the grid by an appropriate number of edges so that all vertices of the grid have degree 4 in $\mathcal{X}^*_{n,m}$. Let $\mathcal{G}_{n,m}$ denote the sandpile group of $\mathcal{X}^*_{n,m}$. Very little is known about the groups $\mathcal{G}_{n,m}$. Dhar et al. [14] found the rank of $\mathcal{G}_{n,n}$ to be n. The rank of $\mathcal{G}_{n,n+1}$ remains a mystery; the rank of $\mathcal{G}_{n,m}$ is at least $\gcd(n+1, m+1) - 1$ [14].

Biggs introduced the Abelian Sandpile Model for undirected graphs into algebraic graph theory under the name "dollar-game" [5, 6], but subsequent work adopted the sandpile terminology [3, 9]. There appears to be a characteristic difference between the choice of ambient spaces between the physics and the algebraic graph theory communities. Physicists prefer finite chunks of discrete infinite homogeneous spaces such as d-dimensional grids with a sink attached to the boundary. Algebraic graph theorists seem to prefer finite homogeneous ambient spaces such as the n-dimensional cube.

There are very few classes of graphs for which the sandpile group structure has been completely determined; these include the complete graphs [23] and the wheel graphs [5]. Only partial results are known for the square grids (Dhar et al. [14], see above). Dartois et al. [11] determine the sandpile group of the Cayley graph of the dihedral group under the standard pair of generators (the n-sided prism); its rank is shown to be 2 or 3 for all $n \geq 3$ but they could not fully resolve the question when exactly is the rank equal to 3.

Bai [3] computed the Sylow p-subgroups for $p > 2$ for the case of the n-dimensional cube and the result was generalized by Jacobson et al. [20] to the Sylow p-subgroups for the Cartesian product of complete graphs $K_{n_1} \times \cdots \times K_{n_s}$ for primes p not dividing $\prod_{i=1}^{s} n_i$.

In [9] it is shown that a planar graph and its dual have isomorphic sandpile groups.

A slight variation of the ASM has been studied in the theory of computing under the name "chip-firing game" [8, 7]. The difference is the absence of the sink in the chip-firing game. This can result in avalanches that never terminate. It was shown (Bjørner et al. [8]) that whether an avalanche terminates depends on the initial state only; so does the final stable state reached. For undirected graphs, a bound on the length of an avalanche is given in terms of the smallest positive eigenvalue of the Laplacian [8].

The directed version of the ASM and chip-firing game were studied in [12, 33] and [7] respectively.

An introduction to self-organized criticality with an indication of its connection to the ASM can be found in [21].

The ASM was originally introduced in statistical physics as a Markov Chain on the space of stable states, with a transition consisting of adding a sandgrain at a random ordinary vertex and stabilizing. Statistical physicists are interested in properties of the stationary distribution, i. e., the uniform distribution over the set of recurrent states, and in the distribution of various random variables associated with the process, such as the length of an avalanche and the number of vertices toppled in an avalanche ("time" and "space"). [1, 13, 19, 27, 32].

Algorithmic aspects of the chip-firing game and ASM are considered in [18, 29, 33, 34].

1.4 Main results: structure of the sandpile monoid

In Chapter 4 we describe the structure of the lattice[4] $\mathcal{L}$ of idempotents of the sandpile monoid in terms of the partially ordered set (poset) of strong components of the ambient space (Theorem 4.2.1); it turns out that $\mathcal{L}$ is isomorphic to the dual semilattice of ideals of the poset of normal[5] strong components of the ambient space $\mathcal{X}^*$. An immediate corollary is that $\mathcal{L}$ is a distributive lattice. It follows via a universal epimorphism $\mathcal{M} \to \mathcal{L}$ that $\mathcal{M}$ is a "distributive lattice of commutative semigroups with a unique idempotent." We also note that conversely, every finite distributive lattice is isomorphic to the semilattice of semigroups of a sandpile monoid.

Uniqueness of the idempotent in a finite semigroup is a particularly strong structural restriction. We characterize the digraphs for which the sandpile semigroup $\mathcal{S}$ has a unique idempotent (Theorem 4.4.7); this, in particular, is the case when the digraph is strongly connected on the ordinary vertices, i. e., when every ordinary vertex is reachable from every ordinary vertex by a directed path through ordinary vertices. If the idempotent $\mathbf{e}$ in the sandpile semigroup $\mathcal{S}$ is unique then the ideal in $\mathcal{S}$ generated by $\mathbf{e}$ is precisely the sandpile group $\mathcal{G}$ and the sandpile semigroup $\mathcal{S}$ is a nilpotent ideal extension of $\mathcal{G}$, i. e., the Rees quotient $\mathcal{S}/\mathcal{G}$ (obtained by contracting $\mathcal{G}$ to a zero element) is nilpotent. We call the Rees quotient $\mathcal{S}/\mathcal{G}$ the *sandpile quotient.*

In Chapters 4 and 5 we demonstrate that structural properties of the sandpile semigroup $\mathcal{S}$ have strong connections to the combinatorial structure of the underlying digraph. Chapter 5 contains our main result in this direction: the classification of rooted digraphs according to the nilpotence class of their sandpile quotient. (A non-nilpotent semigroup has "infinite nilpotence class.") We obtain an asymptotic characterization of the digraphs with sandpile quotients of bounded nilpotence class (near the end of Section 5.1). This characterization in turn has further strong implica-

4. The term "lattice" appears in two meanings in this thesis: a subgroup Λ of finite index in $\mathbb{Z}^n$, and a partially ordered set $\mathcal{L}$ with join and meet. It will always be evident from the context which kind of lattice we mean.

5. We call a strong component *normal* if it contains a cycle. An *abnormal* strong component consists of a single vertex contained in no cycle.

tions on the semigroup structure, and, somewhat surprisingly, on the group structure as well. The main corollary to our classification theorem is the following:

Theorem 1.4.1 *There exist functions ψ_1 and ψ_2 such that if the sandpile quotient $\mathcal{S}/\mathcal{G}$ has nilpotence class k then its order is $|\mathcal{S}/\mathcal{G}| \leq \psi_1(k)$; and the sandpile group $\mathcal{G}$ contains a cyclic subgroup of index $\leq \psi_2(k)$.*

In other words, sandpile semigroups with sandpile quotients of bounded nilpotence class have only a bounded number of elements outside the corresponding sandpile groups, i. e., a bounded number of transient states, and the corresponding sandpile groups are virtually cyclic (cyclic-by-bounded). We note that even under these circumstances, the order of the sandpile group can be exponentially large compared to the number of edges of the underlying digraph (see Corollary 4.4.5).

The work in Chapters 3, 4 and 5, is joint with my advisor László Babai.

1.5 Main results: sandpile group of trees

Since the case of square lattices is practically intractable, we are considering a relatively easier but still rather non-trivial case. We adopt the physicists' approach (see Section 1.3) and consider certain finite subtrees of the infinite d-regular tree $\mathcal{T}(d)$, to which we attach a sink. The augmented space in discussion (denoted as $\mathcal{T}(d,h)$) is constucted as follows: we choose a vertex of $\mathcal{T}(d)$ to be the *root*, take a ball of radius h about the root, add a sink vertex, and connect it to each boundary vertex by $d-1$ edges. We note that statistical properties of this model have been considered by Dhar [13]. We focus on the algebraic structure of the group $G(d,h)$, the sandpile group of $\mathcal{T}(d,h)$ (Chapter 6). We assume $d \geq 3$, $h \geq 1$.

Theorem 1.5.1 *The* rank *of $G(d,h)$ is $(d-1)^h$.*

Theorem 1.5.2 *$G(d,h)$ contains a subgroup isomorphic to $\mathbb{Z}_d^{(d-1)^h}$ ($\mathbb{Z}_d$ stands for the cyclic group of order d). Therefore, for all primes p dividing d, the rank of the Sylow p-subgroup is equal to the rank of $G(d,h)$.*

Definition 1.5.3

(i) Let G be a finite group and H a subgroup of G. H is a *Hall-subgroup* if $\gcd(|H|, |G : H|) = 1$.

(ii) We say that H is a *Hall t-subgroup*[6] of G if H is a Hall-subgroup of G and t and $|H|$ have the same prime divisors.

Note that if G is abelian then for any $t \,\big|\, |G|$, G has a unique Hall t-subgroup.

Theorem 1.5.4 *The Hall $(d-1)$-subgroup of $G(d,h)$ is cyclic of order $(d-1)^h$.*

Definition 1.5.5 Let G be a finite group. The *exponent* of G is the least common multiple of the orders of the elements of G. If G is abelian then the exponent is also the largest order of an element.

Notation 1.5.6

(i) We denote the exponent of $G(d,h)$ by $\exp(d,h)$.

(ii) We define the numbers $\theta(d,n)$ as

$$\theta(d,n) := \frac{(d-1)^n - 1}{d-2}. \tag{1.1}$$

Theorem 1.5.7 *The exponent $\exp(d,h)$ of the group $G(d,h)$ is equal to*

$$(d-1)^h \operatorname{lcm}\{d\theta(d,h+1), \theta(d,h), \theta(d,h-1), \ldots, \theta(d,2)\}. \tag{1.2}$$

This seems to be the first time in the literature that the exponent of a nontrivial class of sandpile groups is found[7]. Using an estimate for $\operatorname{lcm}\{d-1, d^2-1, \ldots, d^h-1\}$,

6. This terminology, while very convenient for our purposes, is not standard. In standard terminology, let π be a set of primes and s be the largest divisor of $|G|$ made up of powers of primes in π. Then a Hall π-subgroup of G is a subgroup of order s. So, our Hall 18-subgroup is a Hall $\{2,3\}$-subgroup in standard terminology.

7. We note that Bai in [3] finds the odd part of the exponent of the sandpile group of the n-cube.

communicated to us by R. Narasimhan (see Theorem 6.6.2), we can log-asymptotically evaluate expression (1.2) and obtain the following corollary:

Corollary 1.5.8 *For every fixed $d \geq 3$, the following asymptotic equality holds as $h \to \infty$:*

$$\log_{d-1} \exp(d,h) \sim \frac{3h^2}{\pi^2}.$$

Next we consider the order of the group $G(d,h)$.

Theorem 1.5.9 *The order of the group $G(d,h)$ is equal to*

$$d(d-1)^h [\theta(d,h+1)]^{d-1} \prod_{n=1}^{h-1} [\theta(d,h+1-n)]^{(d-2)d(d-1)^{n-1}}. \tag{1.3}$$

Corollary 1.5.10 *For every fixed $d \geq 3$, the following asymptotic equality holds as $h \to \infty$:*

$$\log_{d-1} |G(d,h)| \sim c_d (d-1)^h, \tag{1.4}$$

where

$$c_d := d(d-2) \sum_{n=0}^{\infty} (d-1)^{-2-n} \log_{d-1} \frac{(d-1)^{n+2} - 1}{d-2}.$$

This shows that for large h the groups $G(d,h)$ are rather "flat:" the exponent is very small compared to the order. We don't believe this to be the case for instance for the square lattices.

The proofs rest on detailed combinatorial and number-theoretic analysis of the rather complex system of defining relations given by the rows of the Laplacian. For example, for the rank calculation, we find a particular subset of the vertices that turns out to correspond to a basis. We mention one of the lemmata we prove in order to show the upper bound for the exponent.

Lemma 1.5.11 *Let G be a finite abelian group and let $\mathbf{z}_n \in G$ $(0 \leq n \leq t)$. Let $\mathrm{ord}(\mathbf{z}_n) = s$, where $\mathrm{ord}(\mathbf{z}_n)$ denotes the order of $\mathbf{z}_n$ in G. Assume $r_n \mathbf{z}_{n+1} = r_{n+1} \mathbf{z}_n$ $(0 \leq n \leq t-1)$, where $r_n \in \mathbb{Z}$ and $\gcd(r_n, r_{n+1}) = 1$ $(0 \leq n \leq t-1)$. Then $\mathrm{ord}(\mathbf{z}_n) \mid \mathrm{lcm}\{sr_0, r_1, \ldots, r_{n-1}\}$ $(1 \leq n \leq t)$.*

CHAPTER 2
BASIC CONCEPTS

In this chapter we introduce the basic graph-theoretic and semigroup-theoretic terminology.

2.1 Quasi-orders and preorders

We say that a set with a transitive relation $\mathcal{Q} = (\Omega, \prec)$ is a *quasi-order*[1]. Let $\mathcal{Q}' = (\Omega, \preccurlyeq)$ be the reflexive closure of $\mathcal{Q}$, i. e., $x \preccurlyeq y$ if and only if $x \prec y$ or $x = y$. $\mathcal{Q}'$ is a *preorder*, i. e., a reflexive and transitive relation. Let $x, y \in \Omega$. We say that x is *accessible* from y if $x \preccurlyeq y$. Mutual accessibility is an equivalence relation in Ω; its equivalence classes are the *strong components* of the preorder $\mathcal{Q}'$ as well as of the quasi-order $\mathcal{Q}$.

An element y of Ω is called *cyclic* if $y \prec y$. Let $y \in \Omega$ be *acyclic*, i. e., let $y \not\prec y$. We claim that $\{y\}$ is a strong component; indeed if $x \prec y$ and $y \prec x$ for some x, then by transitivity $y \prec y$. We call a strong component *abnormal* if it consists of a single acyclic element. A strong component that is not abnormal is called *normal*. So, a strong component is normal if and only if all its elements are cyclic. Note that it is possible for a normal strong component to have just one element. If $A, B \subseteq \Omega$ then we say that A is *accessible* from B if $(\exists x \in A)(\exists y \in B)$ such that x is accessible from y. Accessibility is a *partial order* on the set of strong components.

Definition 2.1.1

(i) A strong component K is *terminal* if no other strong component is accessible from K.

1. This terminology is not standard. In the literature the terms "quasi-order" and "preorder" are used for a reflexive and transitive relation. Since the term "quasi-order" is used rather infrequently, we took the liberty of borrowing it for a transitive relation.

(ii) The elements of terminal strong components are called *recurrent*.

(iii) Elements of non-terminal strong components are called *transient*.

(iv) A strong component is *initial* if it is not accessible from any other strong component.

(v) A *source* is an acyclic element that is not accessible from any other element. In other words, a source is the unique element of an abnormal initial strong component.

(vi) We call an element in Ω *fully accessible* if it is accessible from every element in Ω.

The following is evident:

Fact 2.1.2 *For a finite quasi-order $\mathcal{Q} = (\Omega, \prec)$, the following are equivalent:*

(i) There exists a fully accessible element in Ω.

(ii) There is a unique terminal equivalence class.

(iii) All recurrent elements in Ω are fully accessible.

Definition 2.1.3 Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order and let $\mathcal{Q}' = (\Omega, \preccurlyeq)$ be its reflexive closure.

(i) Let $I \subseteq \Omega$. I is an *ideal* if $(\forall x \in I)(\forall y \in \Omega)((y \prec x) \Rightarrow (y \in I))$.

(ii) Let $A \subseteq \Omega$. The ideal generated by A is defined as $\{y \in \Omega : (\exists x \in A)(y \preccurlyeq x)\}$.

Notation 2.1.4 For a subset A of Ω, we denote by $I_{\mathcal{Q}}(A)$ the ideal generated by A. We omit the subscript when the quasi-order is clear from the context.

Observation 2.1.5 *Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order.*

(i) Let $A \subseteq \Omega$. The ideal $I(A)$ consists of all elements in Ω accessible from some element in A. In particular, $A \subseteq I(A)$.

(ii) Let $A, B \subseteq \Omega$. Then $I(A \cup B) = I(A) \cup I(B)$.

Definition 2.1.6 Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order.

(i) Let I be an ideal. We say that I is a *normal* ideal, if there are no source vertices in the quasi-order $(I, \prec)$, where $(I, \prec)$ is the restriction of $\mathcal{Q}$ to I. In other words, an ideal is normal, exactly if all initial strong components of the ideal are normal.

(ii) Let $A \subseteq \Omega$. The normal ideal generated by A is the largest normal ideal that is subset of $I(A)$, the ideal generated by A. In other words, it is the ideal generated by the cyclic elements of $I(A)$.

Notation 2.1.7 Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order.

(i) We denote by $c(\mathcal{Q})$ the set of cyclic elements of $\mathcal{Q}$.

(ii) We denote by $\mathcal{Q}^c = (c(\mathcal{Q}), \preccurlyeq)$ the preorder obtained by restricting $\mathcal{Q}$ to $c(\mathcal{Q})$.

(iii) Let $A \subseteq \Omega$. We denote by $\iota_{\mathcal{Q}}(A)$ the normal ideal generated by A. As with $I_{\mathcal{Q}}(A)$ we omit the subscript when the quasi-order is clear from the context.

Observation 2.1.8 *Let* $\mathcal{Q} = (\Omega, \prec)$ *be a quasi-order and let* $A, B \subseteq \Omega$.

(i) $\iota(A) = \iota(I(A))$.

(ii) $\iota(A \cup B) = \iota(A) \cup \iota(B)$.

Notation 2.1.9 Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order.

(i) We denote by $s(\mathcal{Q})$ the set of strong components of $\mathcal{Q}$.

(ii) We denote by $ns(\mathcal{Q})$ the set of normal strong components of $\mathcal{Q}$.

(iii) We denote by $\mathcal{Q}^{ns} = (ns(\mathcal{Q}), \preccurlyeq)$ the partially ordered set of normal strong components of $\mathcal{Q}$.

A *semilattice* is a partially ordered set where every two elements have a meet.

Proposition 2.1.10 *Let $\mathcal{Q} = (\Omega, \prec)$ be a quasi-order. The following semilattices are isomorphic:*

(i) The semilattice of normal ideals of the quasi-order $(\Omega, \prec)$.

(ii) The semilattice of ideals of the preorder $(c(\mathcal{Q}), \preccurlyeq)$.

(iii) The semilattice of ideals of the partial order $(c(\Omega), \preccurlyeq)$.

Proof: Immediate from the definitions. □

2.2 Directed graphs, accessibility preorder

We use the term "digraph" to refer to directed multigraphs. Let $\mathcal{X} = (\Omega, \mathcal{E})$ be a digraph. Ω is the set of vertices (in some contexts called "states") and $\mathcal{E}$ is the set of edges (called "transitions" in the contexts mentioned). By *cycles* we always mean *directed* cycles. We say that vertex y is *accessible* from vertex x if there is a directed path from x to y and that y is *normally accessible* from x if there is a directed path of non-zero length from x to y. Normal accessibility in $\mathcal{X}$ is a quasi-order on Ω. Accessibility is the reflexive closure (see Section 2.1) of normal accessibility. We will refer to normal accessibility as the *accessibility quasi-order* and to accessibility as the *accessibility preorder*.

Normal strong components in the accessibility preorder are those strong components which contain a cycle.

We adopt the terms "recurrent," "transient," etc from Definitions 2.1.1 to digraphs to mean recurrent, transient, etc with respect to the *accessibility quasi-order*. For finite digraphs, the terms "recurrent" and "transient" are consistent with the usage in finite Markov chains. A *source* in a directed graph is a vertex of indegree 0.

Notation 2.2.1 Let $\mathcal{X} = (\Omega, \mathcal{E})$ be a directed graph and let $A \subseteq \Omega$.

(i) We denote by $\mathcal{Q}(\mathcal{X}) = (\Omega, \prec)$ the accessibility quasi-order in $\mathcal{X}$.

(ii) We denote by $\mathcal{P}(\mathcal{X}) = (\Omega, \preccurlyeq)$ the accessibility preorder in $\mathcal{X}$. Note that $\mathcal{P}(\mathcal{X}) = \mathcal{Q}(\mathcal{X})'$.

(iii) We denote by $ns(\mathcal{X})$ the set of normal strong components in $\mathcal{Q}(\mathcal{X})$.

(iv) We denote by $\mathcal{Q}(\mathcal{X})^{ns} = (ns(\mathcal{X}), \preccurlyeq)$ the partial order on the set of the normal strong components of $\mathcal{X}$.

(v) We denote by $\mathcal{X}[A]$ the subgraph of $\mathcal{X}$ induced on A.

2.3 Semigroups

Our standard reference for semigroups is [16]. A *semigroup* is a set with an associative operation. A *monoid* is a semigroup with identity.

Definition 2.3.1 If a semigroup $\mathcal{S}$ does not have an identity element, we set $\mathcal{S}^1 = \mathcal{S} \cup \{1\}$ to denote the semigroup $\mathcal{S}$ with an identity element added. If $\mathcal{S}$ does have an identity then we set $\mathcal{S}^1 = \mathcal{S}$. We call $\mathcal{S}^1$ the *monoid extension* of the semigroup $\mathcal{S}$.

We now consider a natural one-sided inverse of this operation.

Definition 2.3.2 For a monoid $\mathcal{M}$, let $\mathcal{M}^-$ denote the subsemigroup of $\mathcal{M}$ generated by the set $\mathcal{M} \setminus \{1\}$. We call $\mathcal{M}^-$ the *semigroup reduction* of the monoid.

Observe that $|\mathcal{M} \setminus \mathcal{M}^-| \leq 1$ and $(\mathcal{M}^-)^1 = \mathcal{M}$. The converse, $(\mathcal{S}^1)^- = \mathcal{S}$, is not necessarily true but it is true if $\mathcal{S}$ is not a monoid.

A subset $I \subseteq \mathcal{S}$ is an *ideal* if $\mathcal{S}^1 I \mathcal{S}^1 \subseteq I$. The *principal ideal* generated by $y \in \mathcal{S}$ is the set $\mathcal{S}^1 y \mathcal{S}^1$. An element $x \in \mathcal{S}$ is *accessible* from an element $y \in \mathcal{S}$ if $x \in \mathcal{S}^1 y \mathcal{S}^1$. We say that an element $x \in \mathcal{S}$ is *fully accessible* if it is accessible from every element of $\mathcal{S}$. (All semigroup-theoretic terms except for "full accessibility" and "semigroup reduction" are standard.) Note that if $\mathcal{S}$ is not empty then $x \in \mathcal{S}$ is fully accessible in $\mathcal{S}$ if and only if it is fully accessible in $\mathcal{S}^1$.

A *minimal ideal* is a nonempty ideal which does not properly contain any nonempty ideal.

If I and J are ideals then IJ is also an ideal and $IJ \subseteq I \cap J$; therefore the intersection of a finite number of nonempty ideals is nonempty. In particular, a semigroup has at most one minimal ideal; a finite nonempty semigroup has exactly one minimal ideal.

Definition 2.3.3 The *kernel* $\mathcal{G}(\mathcal{S})$ of the semigroup $\mathcal{S}$ is the intersection of all nonempty ideals of $\mathcal{S}$.

Note that the kernel is the unique minimal ideal of $\mathcal{S}$ if $\mathcal{S}$ has a minimal ideal; otherwise the kernel is empty. Note also that if $\mathcal{S}$ is not empty then $\mathcal{G}(\mathcal{S}) = \mathcal{G}(\mathcal{S}^1)$.

We require the following well-known facts.

Fact 2.3.4

(i) The kernel of a semigroup consists of the fully accessible elements of the semigroup.

(ii) The kernel of a nonempty finite semigroup is not empty. In other words, a nonempty finite semigroup has a unique minimal ideal.

(iii) A minimal ideal of a commutative semigroup is a group.

Proof: (i) The elements accessible from x form the principal ideal $\mathcal{S}^1 x \mathcal{S}^1$. Clearly, the kernel is the intersection of all principal ideals. (ii) (Cf. [16], p. 44.) Take the intersection of all nonempty ideals. (iii) In a minimal ideal, elements are mutually accessible. In a commutative semigroup this means one has "division." □

Corollary 2.3.5 *Let $\mathcal{S}$ be a nonempty finite commutative semigroup. Then $\mathcal{S}$ has a unique minimal ideal, $\mathcal{G}(\mathcal{S})$, the kernel of $\mathcal{S}$, consisting of all fully accessible elements of $\mathcal{S}$. The kernel is a group.*

The significance of this observation to our main subject is that the sandpile monoid is a nonempty finite commutative semigroup; therefore its kernel is a group; it consists of the fully accessible elements of the monoid (called "recurrent elements" in the sandpile context). This group is called the "sandpile group." Note that the sandpile group is determined by the abstract structure of the sandpile monoid and does not depend on the underlying "ambient space" (digraph).

2.4 Cayley digraphs

The *right ideal* generated by $y \in \mathcal{S}$ is the set $y\mathcal{S}^1$. We say that $x \in \mathcal{S}$ is *right accessible* from $y \in \mathcal{S}$ if $x \in y\mathcal{S}^1$. In a commutative semigroup, right accessibility is equivalent to accessibility as defined in the previous section.

Definition 2.4.1 Let $\mathcal{S}$ be a semigroup and let T be a set of generators of $\mathcal{S}$. The *(right) Cayley digraph* $\Gamma(\mathcal{S}, T)$ has $\mathcal{S}$ for its set of vertices and has edges $s \to st$, where $s \in \mathcal{S}$, $t \in T$.

When we refer to digraph properties in the context of a semigroup with a given set of generators, we refer to the corresponding (right) Cayley digraph unless explicitly stated otherwise.

Right accessibility is identical with accessibility in the right Cayley digraph $\Gamma(\mathcal{S}, T)$ regardless of the choice of the set T of generators. Therefore the associated preorder does not depend on the choice of generators. In particular, the strong components don't depend on the choice of generators. The preorder on $\mathcal{S}$ corresponding to right accessibility are called *Green's right preorder*; the corresponding strong components will be called *Green's right classes.* The digraph concepts carry over to this context; in particular, we can speak of *terminal right Green classes.* It is clear that *a terminal right Green class is the right ideal generated by any of its elements* and therefore it is a minimal right ideal. Combined with Corollary 2.3.5, we obtain the following:

Proposition 2.4.2 *Let $\mathcal{S}$ be a nonempty finite commutative semigroup. Then $\mathcal{S}$ has a unique terminal Green class, namely the kernel, $\mathcal{G}(\mathcal{S})$. Consequently, the Cayley digraphs of a nonempty finite commutative semigroup satisfy the equivalent conditions of Fact 2.1.2.*

2.5 Unique idempotents

An *idempotent* is an element x such that $xx = x$. Every nonempty finite semigroup contains an idempotent.

Note that a *semilattice* is a commutative semigroup of which all elements are idempotent.

The idempotents of a commutative semigroup form a semilattice.

Having a unique idempotent is a particularly strong structural constraint on a semigroup.

Definition 2.5.1 Let $\mathcal{S}$ be a semigroup and $I \subset \mathcal{S}$ an ideal. The *Rees congruence* (denoted by $\sim_R$) is defined as follows: $a \sim_R b$ if $a = b$ or $a, b \in I$ (cf. [16] p. 16). The *Rees quotient* $\mathcal{S}/I$ is a semigroup with zero. We say that $\mathcal{S}$ is an *ideal extension* of the semigroup I by $\mathcal{S}/I$, a semigroup with zero (cf. [16] p. 54). We say that $\mathcal{S}$ is a *nilpotent ideal extension* of I by $\mathcal{S}/I$ if the Rees quotient $\mathcal{S}/I$ is nilpotent.

The *nilpotence class* of a nilpotent semigroup $\mathcal{N}$ with zero is the smallest k such that $\mathcal{N}^k = \{0\}$.

Detailed structure of the nilpotent commutative finite semigroups is given in [17].

Proposition 2.5.2 *For a nonempty finite commutative semigroup $\mathcal{S}$ generated by a set T, the following are equivalent:*
(i) Every element of $\mathcal{S}$ has a power which is fully accessible.
(ii) Every element of T has a power which is fully accessible.
(iii) The Rees quotient of $\mathcal{S}$ by its kernel $\mathcal{G}(\mathcal{S})$ is nilpotent, i. e., $\mathcal{S}$ is a nilpotent ideal extension of a group.
(iv) $\mathcal{S}$ has a unique idempotent.

Proof: Clearly, (i) implies (ii). To see that (ii) implies (iii), recall that the kernel $\mathcal{G} = \mathcal{G}(\mathcal{S})$ consists of the fully accessible elements. So in the Rees quotient every generator is nilpotent. By commutativity it follows that every element of $\mathcal{S}/\mathcal{G}$ is nilpotent. Finally, we claim that every product of length $n+1$ in $\mathcal{S}/\mathcal{G}$ is is zero, where $n = |\mathcal{S}/\mathcal{G}|$. Take such a product, $s_0 \dots s_n$ and consider the prefixes $p_i = s_0 \dots s_i$. By the pigeon hole principle, $p_i = p_j$ for some $i < j$. But $p_j = p_i x$ for some $x \in \mathcal{S}/\mathcal{G}$, so by induction $p_i = p_i x^r$ for all $r \geq 0$. Some power x^t is zero hence $p_i = p_i \cdot 0 = 0$ and thus $p_n = p_i y = 0$. Assume (iii) now and let x be an idempotent. Now, by the nilpotence of $\mathcal{S}/\mathcal{G}$, some power of x belongs to $\mathcal{G}$ and therefore x belongs to $\mathcal{G}$. But $\mathcal{G}$ is a group, so x can only be the identity element of $\mathcal{G}$, proving (iv). Finally, let us assume (iv); let y be the unique idempotent in $\mathcal{S}$. Since every element of $\mathcal{S}$ has a power which is an idempotent, y is fully accessible, proving (i). □

CHAPTER 3
THE ABELIAN SANDPILE MODEL FOR DIRECTED GRAPHS

In Sections 3.1 and 3.2 we introduce notation to further formalize the concepts discussed in the Introduction. While our definitions are equivalent to those previously appearing in the literature, our treatment seems novel and is intended to give a unified introduction with greater conceptual clarity by emphasizing the role played by the simple facts of semigroup theory discussed in the preceding chapter.

3.1 The Sandpile Monoid, Semigroup and Group

Definition 3.1.1 An *ambient space* is a rooted directed multigraph ("multidigraph") $\mathcal{X}^* = (V^*, E^*)$ with a vertex designated as the *sink*. The sink is accessible from every vertex.

The vertices other than the sink will be called *ordinary*. Let V denote the set of ordinary vertices and $\mathcal{X} = (V, E)$ the subgraph of $\mathcal{X}^*$ induced on V. For $u \in V$, we write $\deg(u)$ to denote the out-degree of u in $\mathcal{X}$ and $\deg_*(u)$ to denote the out-degree of u in $\mathcal{X}^*$. (So $\deg_*(u) \geq \deg(u)$ and $\deg_*(u) \geq 1$ for every ordinary vertex u.)

Definition 3.1.2 A vertex $v \in V^*$ is *irrelevant* if $v \in V$ and $\deg_*(v) = 1$; otherwise v is *relevant*. In particular, the sink is always relevant. An edge $v \to u$ ($v, u \in V^*$) is *irrelevant* if v is irrelevant; all other edges are *relevant*. In particular, if v is the sink then any edge $v \to u$ is relevant.

Note that relevant ordinary vertices satisfy $\deg_* \geq 2$.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$. The *state space* is $\mathbb{N}^V$; our notation for the states is $\mathbf{h} = (h_u)_{u \in V}$ ($h_u \geq 0$). We call h_u the *height* of u in state $\mathbf{h}$. In a state $\mathbf{h}$, the vertex

$u \in V$ is *stable* if $h_u < \deg_*(u)$. A state is *stable* if all vertices are stable. $\mathcal{M}$ denotes the set of stable states. Observe that

$$|\mathcal{M}| = \prod_{u \in V} \deg_*(u). \tag{3.1}$$

Let A denote the *adjacency matrix* of $\mathcal{X}^*$, i. e., the $|V^*| \times |V^*|$ matrix with entries a_{vu} $(v, u \in V^*)$ defined as the number of $v \to u$ edges. Let $\mathbf{h} \in \mathbb{N}^V$ be a state. If v is an unstable vertex, i. e., $h_v \geq \deg_*(v)$, we can apply the *toppling operator* α_v to $\mathbf{h}$; the result is $\alpha_v(\mathbf{h}) = (h'_u)_{u \in V}$ where $h'_v = h_v - \deg_*(v) + a_{vv}$ and $h'_u = h_u + a_{vu}$ for $u \neq v$. To *stabilize* $\mathbf{h}$ means to topple unstable vertices in succession until a stable state $\sigma(\mathbf{h})$ is reached. The function $\sigma : \mathbb{N}^V \to \mathcal{M}$ is well-defined (see the reference to the "Jordan–Hölder argument" in the Introduction). We define addition on $\mathcal{M}$ as follows: for $\mathbf{h}_1, \mathbf{h}_2 \in \mathcal{M}$, let

$$\mathbf{h}_1 \oplus \mathbf{h}_2 := \sigma(\mathbf{h}_1 + \mathbf{h}_2). \tag{3.2}$$

Next we define the three algebraic structures central to the theory.

Definition 3.1.3 The **sandpile monoid** of $\mathcal{X}^*$ is the set $\mathcal{M} = \mathcal{M}(\mathcal{X}^*)$ of stable states under the addition operation defined by equation (3.2).

So $\mathcal{M}$ is a commutative monoid.

Definition 3.1.4 The **sandpile semigroup** $\mathcal{S} = \mathcal{S}(\mathcal{X}^*)$ is the semigroup reduction of the sandpile monoid $\mathcal{M}(\mathcal{X}^*)$ (see Definition 2.3.2).

In other words, $\mathcal{S}$ is the semigroup generated by the non-zero elements of $\mathcal{M}$.

Definition 3.1.5 The **sandpile group** $\mathcal{G} = \mathcal{G}(\mathcal{X}^*)$ is the kernel of the sandpile monoid (see Definition 2.3.3).

In other words (cf. Fact 2.3.4), $\mathcal{G}(\mathcal{X}^*)$ is the unique minimal ideal of $\mathcal{M}(\mathcal{X}^*)$. It consists of the fully accessible elements of $\mathcal{M}(\mathcal{X}^*)$; these are exactly the *recurrent states* as defined in the Introduction.

Let $\mathbf{h}_{\max}$ denote the saturated stable state, i. e., $(\forall v \in V)(\mathbf{h}_{\max}(v) := \deg_*(v)-1)$. Clearly, $\mathbf{h}_{\max}$ is accessible from every state, so $\mathbf{h}_{\max}$ is recurrent (belongs to the sandpile group). We shall use the following observation.

Observation 3.1.6 *A state is recurrent if and only if it is accessible from* $\mathbf{h}_{\max}$.

3.2 Standard generators, relations, the Laplacian

Let $B \subseteq V$ and $\mathbf{h} \in \mathbb{N}^V$. We define the *weight* of $\mathbf{h}$ on B as

$$w_B(\mathbf{h}) = \sum_{v \in B} h_v. \tag{3.3}$$

The *weight* of $\mathbf{h}$ is $w(\mathbf{h}) := w_V(\mathbf{h})$.

Note that the sandpile semigroup $\mathcal{S}$ is the subsemigroup of the sandpile monoid $\mathcal{M}$ generated by the states of weight 1 and that

$$\mathcal{M} = \mathcal{S}^1 = \mathcal{S} \cup \{0\}. \tag{3.4}$$

We also note that 0 may or may not belong to $\mathcal{S}$; we shall characterize the cases when $0 \in \mathcal{S}$ in terms of properties of the ambient space $\mathcal{X}^*$ (Observation 4.2.1 and Proposition 4.2.5).

Definition 3.2.1

(i) We denote by $\mathbf{t}_u$ the weight-1 state which has its single grain on $u \in V$.

For this to be a stable state, u must be relevant (Def. 3.1.2). So $\mathcal{S}$ is the semigroup generated by the $\mathbf{t}_u$ for relevant vertices u.

(ii) The states $\mathbf{t}_u$ for relevant ordinary vertices u will be called the *standard generators* of $\mathcal{S}$.

Observation 3.2.2

(i) Let $\mathbf{h} \in \mathbb{N}^V$ *be a state with* $\mathbf{h}_v \geq \deg_*(v)$. *Then* $\mathbf{h} - \deg_*(u)\mathbf{t}_u + \sum_{v \in V} \mathbf{t}_v$ *is the state obtained from* $\mathbf{h}$ *by toppling the unstable vertex* v.

(ii) For all $v \in V$*, the following relation holds in* $\mathcal{M}$*:*

$$\deg_*(v)\mathbf{t}_v = \sum_{u \in V} a_{v,u}\mathbf{t}_u. \tag{3.5}$$

(With some abuse of notation, we omitted σ *from each side of equation (3.5). Technically, we should write* $\sigma((\deg_*(v)\mathbf{t}_v) = \sigma(\sum_{u \in V} a_{v,u}\mathbf{t}_u)$*. Note that for an irrelevant vertex* v*, relation (3.5) turns into* $\mathbf{t}_v = 0$ *as it should.)*

Definition 3.2.3 For every ordinary vertex v, let x_v be a symbol associated with v and consider the set of relations $\mathcal{R} := \{\deg_*(v)x_v = \sum_{u \in V} a_{v,u}x_u : v \in V\}$.

(i) Let $\hat{\mathcal{M}}$ be the commutative monoid generated by $\{x_v : v \in V\}$ subject to the set of defining relations $\mathcal{R}$.

(ii) Let $\hat{\mathcal{S}}$ be the commutative semigroup generated by $\{x_v : v \in V\}$ subject to the set of defining relations $\mathcal{R}$.

(iii) Let $\hat{\mathcal{G}}$ be the abelian group generated by $\{x_v : v \in V\}$ subject to the set of defining relations $\mathcal{R}$.

Proposition 3.2.4 *The sandpile monoid* $\mathcal{M}$ *is isomorphic to* $\hat{\mathcal{M}}$*.*

Proof: $\mathcal{M}$ is generated by $\{\mathbf{t}_v : v \in V\}$ and the $\mathbf{t}_v$ satisfy the relations (3.5). Therefore there is an epimorphism from $\hat{\mathcal{M}}$ onto $\mathcal{M}$. We show that $|\hat{\mathcal{M}}| \leq |\mathcal{M}|$, i. e., $|\hat{\mathcal{M}}| \leq \prod_{v \in V} \deg_*(v)$ (see equation 3.1). Indeed, every element of $\hat{\mathcal{M}}$ can be written in the canonical form

$$\sum_{v \in V} k_v x_v, \text{ where, } 0 \leq k_v < \deg_*(v). \tag{3.6}$$

□

The following construction will play a role in Section 5.3.

Definition 3.2.5 Let $I \subseteq V$ be an ideal in $\mathcal{X}$ in the accessibility preorder. We define the *contraction* of $\mathcal{X}^*$ by I to be the ambient space $\mathcal{X}^*/I$ having vertex set $V^* \setminus I$ and edges defined as follows: Let the contraction map $f : V^* \to V^* \setminus I$ be defined by $f(v) = v_0$ for $v \in I$ where v_0 is the sink of $\mathcal{X}^*$; $f(v) = v$ otherwise. With every edge $v \to u$ in $\mathcal{X}^*$ we associate an edge $f(v) \to f(u)$ in $\mathcal{X}^*/I$. The sink of $\mathcal{X}^*/I$ remains v_0.

Corollary 3.2.6 *Let $I \subseteq V$ be an ideal in $\mathcal{X}$ in the accessibility preorder. Let $\mathcal{R}(I)$ be the set of relations $\{x_v = 0 \,:\, v \in I\}$. Then $\mathcal{M}(\mathcal{X}^*/I) = \mathcal{M}/\mathcal{R}(I)$.*

Proof: Clear. □

Proposition 3.2.7 *The sandpile semigroup $\mathcal{S}$ is isomorphic to $\hat{\mathcal{S}}$.*

Proof: As with the sandpile monoid, the sandpile semigroup $\mathcal{S}$ is generated by $\{\mathbf{t}_v : v \in V\}$ and the $\mathbf{t}_v$ satisfy the relations (3.5). Therefore there is an epimorphism from $\hat{\mathcal{S}}$ onto $\mathcal{S}$. In particular, $|\mathcal{S}| \leq |\hat{\mathcal{S}}|$.

We need to show that $|\mathcal{S}| \geq |\hat{\mathcal{S}}|$.

First, observe that if $1 \in \hat{\mathcal{S}}$ then $1 \in \mathcal{S}$, since $\mathcal{S}$ is a homomorphic image of $\hat{\mathcal{S}}$. Therefore

$$|\mathcal{S}^1| \leq |\hat{\mathcal{S}}^1| \tag{3.7}$$

and

$$|\mathcal{S}^1| - |\mathcal{S}| \leq |\hat{\mathcal{S}}^1| - |\hat{\mathcal{S}}|. \tag{3.8}$$

We now have

$$|\mathcal{M}| = |\mathcal{S}^1| \leq |\hat{\mathcal{S}}^1| = |\hat{\mathcal{M}}| = |\mathcal{M}|, \tag{3.9}$$

where we have used inequaliry (3.7) and Proposition 3.2.4. Equation (3.9) yields

$$|\mathcal{S}^1| = |\hat{\mathcal{S}}^1|. \tag{3.10}$$

Combining equations (3.8) and (3.10) we obtain $|\mathcal{S}| \geq |\hat{\mathcal{S}}|$. □

Definition 3.2.8 Let $\mathcal{M}$ be a monoid, let $\mathcal{G}$ be a group and let $\phi : \mathcal{M} \to \mathcal{G}$ be a homomorphism. We say that $(\phi, \mathcal{G})$ is the *universal group* of $\mathcal{M}$ if every homomorphism from $\mathcal{M}$ to a group factors through ϕ.

Observation 3.2.9 *If $\langle W|R\rangle$ is a presentation of a monoid $\mathcal{M}$, then $\langle W|R\rangle$ is also a presentation of the universal group $\mathcal{G}$ of $\mathcal{M}$ as a group.*

Observation 3.2.10 *Let $\mathcal{M}$ be a finite commutative monoid and let $\mathcal{G}$ be the minimal ideal of $\mathcal{M}$. Let $e \in \mathcal{G}$ be the identity in $\mathcal{G}$ and let $\phi : \mathcal{M} \to \mathcal{G}$ be defined by $\phi(x) := e + x$. Then $(\phi, \mathcal{G})$ is the universal group of $\mathcal{M}$.*

Proof: Let $\psi : \mathcal{M} \to \mathcal{G}_1$ be a homomorphism, where $\mathcal{G}_1$ is a group. Let e_1 be the identity in $\mathcal{G}_1$. First we note that $\phi(e) = e_1$ because e_1 is the only idempotent in $\mathcal{G}_1$. Now define the homomorphism $\omega : \mathcal{G} \to \mathcal{G}_1$ as the restriction of ψ to $\mathcal{G}$. For $x \in \mathcal{M}$, we have $\omega \circ \phi(x) = \omega(\phi(x)) = \omega(x+e) = \psi(x+e) = \psi(x)+\psi(e) = \psi(x)+e_1 = \psi(x)$. □

Corollary 3.2.11 *The sandpile group is the universal group of the sandpile monoid (under the homomorphism described in Fact 3.2.10).*

Corollary 3.2.12 *The sandpile group $\mathcal{G}$ is isomorphic to $\hat{\mathcal{G}}$.*

The following corollary will be used in the proof of Corollary 5.3.2.

Corollary 3.2.13 *Let $I \subseteq V$ be an ideal in $\mathcal{X}$ in the accessibility preorder. Let $\mathcal{R}(I)$ be the set of relations $\{x_v = 0 \,:\, v \in I\}$. Then $\mathcal{G}(\mathcal{X}^*/I) = \mathcal{G}/\mathcal{R}(I)$.*

Proof: Clear. □

Definition 3.2.14 The *Laplacian* $L = (L_{vu})_{v,u \in V^*}$ of $\mathcal{X}^*$ is defined by

$$L_{vu} := \begin{cases} \deg_*(v) - a_{vv} & \text{if } v = u, \\ -a_{vu} & \text{otherwise.} \end{cases} \tag{3.11}$$

Note that the Laplacian does not change if we add loops at vertices.

Definition 3.2.15

(i) The *reduced Laplacian* $\Delta = (\Delta_{vu})_{v,u \in V}$ of $\mathcal{X}^*$ is defined as the matrix obtained from the Laplacian L be deleting the row and column corresponding to the sink.

(ii) Let Δ_v denote the row of the reduced Laplacian Δ corresponding to vertex v.

(iii) Let $\Lambda := \sum_{v \in V} \mathbb{Z}\Delta_v$ be the lattice in $\mathbb{Z}^V$ spanned by the rows of the reduced Laplacian.

Corollary 3.2.16 *The sandpile group is generated by the symbols* $\{x_v \ : \ v \in V\}$ *subject to the set of defining relations* $\mathcal{R} := \{\Delta_{vv}x_v = \sum_{u \in V} \Delta_{v,u}x_u : v \in V\}$.

Note that the reduced Laplacian Δ (and therefore the sandpile group $\mathcal{G}$) does not change if we add loops to the ambient space $\mathcal{X}^*$.

Corollary 3.2.17 *The sandpile group is isomorphic to the quotient* $\mathbb{Z}^V/\Lambda$.

Corollary 3.2.18 (Dhar[12]) *The order of the sandpile group* $\mathcal{G}$ *is the number of directed spanning trees of* $\mathcal{X}^*$*, directed toward the root.*

Proof: By Corollary 3.2.17 we have $|\mathcal{G}| = \det(\Delta)$. The result now follows from the directed graph version of Kirchhoff's [22] classical Matrix-Tree Theorem (see Tutte [36]). □

Directed spanning trees directed away from the root are also called *arborescences* (cf. Lovász [24], Problem 4.16, p. 36).

CHAPTER 4
IDEMPOTENTS IN THE SANDPILE MONOID

4.1 Basic observations

Definition 4.1.1 Let $\mathcal{Y}$ be a digraph and let $D \subseteq V(\mathcal{Y})$. We call $\mathcal{Y}$ a *functional* digraph with domain D if each vertex in D has out-degree 1 in $\mathcal{Y}$ and all other vertices have out-degree 0.

If $\mathcal{Y} = \mathcal{X}^*$ for some ambient space $\mathcal{X}^*$, then by *functional subgraph* of $\mathcal{X}^*$ we mean a subgraph of $\mathcal{X}^*$ that is functional with domain V. (We exclude the sink from the domain.)

Observation 4.1.2 *For a functional subgraph $\mathcal{Y}$ of $\mathcal{X}^*$, the following are equivalent:*

(i) $\mathcal{Y}$ is a directed spanning tree directed toward the sink.

(ii) Disregarding orientations, $\mathcal{Y}$ is a spanning tree of $\mathcal{X}^*$.

(iii) $\mathcal{Y}$ is acyclic.

Let $\mathcal{T}$ denote the set of stable transient states, i. e., $\mathcal{T} = \mathcal{M} \setminus \mathcal{G}$. The following is immediate from equation (3.1) and Theorem 3.2.18, since by hypothesis the sink is accessible from every vertex.

Corollary 4.1.3

(i) $|\mathcal{M}|$ is the number of functional spanning subgraphs of $\mathcal{X}^*$.

(ii) $|\mathcal{T}|$ is the number of those functional spanning subgraphs of $\mathcal{X}^*$ which contain a cycle.

Corollary 4.1.4

$$|\mathcal{T}| \geq \prod_{v \in V} \deg(v). \tag{4.1}$$

Proof: $\prod_{v \in V} \deg(v)$ is the number of those functional subgraphs which have no edge pointing to the sink; no such subgraph is a spanning tree of $\mathcal{X}^*$. □

Definition 4.1.5

(i) Given a state $\mathbf{h} \in \mathbb{N}^V$, we say that a subset $B \subseteq V(\mathcal{X})$ is *blank* if $w_B(\mathbf{h}) = 0$ (see eqn. 3.3).

(ii) We say that a subgraph is *blank* if the vertex set of the subgraph is blank.

Lemma 4.1.6 *Let C be a cycle in $\mathcal{X}$. If C is blank with respect to state $\mathbf{h}$ but not blank with respect to state $\mathbf{h}'$ then $\mathbf{h}$ is not accessible from $\mathbf{h}'$. In particular, a recurrent state cannot have a blank cycle.*

Proof: Every cycle in $\mathcal{X}$ has at least one vertex of degree $\deg^*(v) \geq 2$ because of the condition that the root is accessible from every vertex, so the saturated state has at least one grain on C. If C is not blank, toppling a vertex cannot leave C blank. □

Corollary 4.1.7 *If the zero state is accessible from a state $\mathbf{h}$ then $h_u = 0$ for all vertices from which a cycle in $\mathcal{X}$ is accessible.*

Definition 4.1.8 A state $\mathbf{h}$ is *semistable* if for all vertices $v \in V$, $h_v \leq 2\deg_*(v) - 1$. Semistable stabilization is a toppling sequence where toppling at $v \in V$ can occur only when $h_v \geq 2\deg^*(v)$.

Semistable stabilization is useful in showing that a state is recurrent.

Lemma 4.1.9 *Let $u \in V$. Then there exists a semistable state $\mathbf{h}$ and $k \in \mathbb{N}$ such that $\sigma(\mathbf{h}) = \sigma(k\mathbf{t}_u)$ and $h_v \geq \deg_*(v) - 1$ for all vertices v accessible from u.*

Proof: Taking multiples of $\mathbf{t}_u$ means adding grains at u. If we apply semistable stabilization, the grains will eventually swamp all vertices accessible from u; semistable toppling of v will never reduce the height below $\deg_*(v) - 1$. □

4.2 Monoid structure versus ambient space structure

In this section we establish basic structural connections between the ambient space and the sandpile monoid, semigroup and group. We fix the ambient space $\mathcal{X}^*$; the symbols $\mathcal{M}$, $\mathcal{G}$, $\mathcal{S}$ refer to $\mathcal{M}(\mathcal{X}^*)$, $\mathcal{G}(\mathcal{X}^*)$, and $\mathcal{S}(\mathcal{X}^*)$, respectively.

First we observe, as a consequence of equation (3.1):

Observation 4.2.1 *The following are equivalent:*

(i) $\mathcal{S} = \emptyset$*;*

(ii) $\mathcal{M} = \{0\}$*;*

(iii) all ordinary vertices are irrelevant (Def. 3.1.2);

(iv) $\mathcal{X}^*$ *is a directed tree.*

Definition 4.2.2 Let u be a vertex of $\mathcal{X}$. We say that u is a *point of no return* if no cycle (and therefore no normal strong component) of $\mathcal{X}$ is accessible from u.

Note that $\mathcal{M} = \mathcal{S}$ if and only if $0 \in \mathcal{S}$. Next we characterize the ambient spaces with this property.

Definition 4.2.3 Let $\mathbf{h} \in \mathbb{N}^V$ be a state. The *support* of $\mathbf{h}$, denoted by $\operatorname{supp}(\mathbf{h})$, is defined to be the set of vertices of $\mathcal{X}$ on which $\mathbf{h}$ is non-zero, i. e., $\operatorname{supp}(\mathbf{h}) := \{v \in V : \mathbf{h}(v) > 0\}$.

$I(\operatorname{supp}(\mathbf{x}))$ denotes the ideal generated by $\operatorname{supp}(\mathbf{x})$ in the accessibility quasi-order $\mathcal{Q}(\mathcal{X}^*) = (V, \prec)$ (see Definition 2.1.3 and Notation 2.2.1).

Lemma 4.2.4 (Flushing a DAG) *Let* $\mathbf{h} \in \mathbb{N}^V$ *be a state concentrated on the points of no return. Set* $k := \prod_{v \in I(\operatorname{supp}(\mathbf{h}))} \deg_*(v)$*. Then* $\sigma(k\mathbf{h}) = 0$.

Proof: The induced subgraph $\mathcal{X}[I(\text{supp}(\mathbf{h})]$ is a DAG. Let $v_1, \ldots, v_n$ be the source vertices (vertices of in-degree 0) of the induced subgraph $\mathcal{X}[I(\text{supp}(\mathbf{h})]$. Then $\sigma(\prod_{i=1}^{n} \deg_*(v_i)\mathbf{h})$ is blank on $\{v_1, \ldots, v_n\}$. The result follows by induction. □

Proposition 4.2.5 $\mathcal{M} = \mathcal{S} \neq \{0\}$ *if and only if there is a relevant point of no return in* $\mathcal{X}$*.*

Proof: Assume $\mathcal{M} = \mathcal{S}$, i. e., $0 \in \mathcal{S}$. This means that 0 is accessible from some non-zero stable state $\mathbf{h}$. By Corollary 4.1.7, the weight of $\mathbf{h}$ is concentrated on the points of no return. One of these, then, must be relevant since irrelevant vertices cannot carry grains. Conversely, let u be a relevant point of no return. By Lemma 4.2.4, we have that 0 is accessible from the weight-1 state $\mathbf{t}_u$. □

Next we study the case when $\mathcal{S} = \mathcal{G}$. "Directed acyclic graph" is abbreviated as "DAG."

Proposition 4.2.6 $\mathcal{M} = \mathcal{G}$ *if and only if* $\mathcal{X}$ *is a DAG.*

Proof: The question is, when is 0 accessible from $\mathbf{h}_{\max}$. If $\mathcal{X}$ is a DAG, we can sort the vertices topologically (all arrows go downward) and empty $\mathbf{h}_{\max}$ by adding grains and toppling each vertex, starting at the top. The necessity is immediate from Lemma 4.1.6. □

Corollary 4.2.7 *S* $\mathcal{M} = \mathcal{S} = \mathcal{G}$ *if and only if* $\mathcal{X}$ *is a DAG and there is a vertex* v *such that* $\deg_*(v) > 1$.

Proof: We have that $\mathcal{G} \subseteq \mathcal{S}$ if and only if there is a vertex v such that $\deg_*(v) > 1$. □

Proposition 4.2.8 $\mathcal{M} \neq \mathcal{S} = \mathcal{G}$ *if and only if* $\mathcal{X}$ *contains a unique cycle and all vertices not on this cycle are irrelevant.*

Proof: We have that $\mathcal{M} \neq \mathcal{S} = \mathcal{G}$ if and only if 0 is the only stable state that is not recurrent. If 0 is the only stable state that is not recurrent, then there is exactly one functional spanning subgraph in $\mathcal{X}$ which is not a spanning tree directed to the root (Corollary 4.1.3 and Theorem 3.2.18). This is easily seen to be equivalent to the structure described. □

Corollary 4.2.9 *In the undirected wheel graph on $n+1$ vertices, all $2^n - 1$ non-zero stable states are recurrent.*

The above serves an example of an instance where the sandpile group is cyclic and its order is exponentially large compared to the number of edges of the underlying directed graph (see comments at the end of Section 1.4).

Proposition 4.2.10 *If $\mathcal{M} \neq \mathcal{S} = \mathcal{G}$ then $\mathcal{G}$ is a cyclic group. The identity element of the group is the saturated state $\mathbf{h}_{\max}$.*

Proof: Let u be a vertex on the unique cycle. Then $\mathbf{t}_u$ clearly generates $\mathbf{t}_{u+1}$ where "$u+1$" refers to the next vertex along the cycle. □

4.3 The semilattice of idempotents

Definition 4.3.1 Let $\mathcal{S}$ be a semigroup, let $\mathcal{L}$ be a semilattice, and let $\phi : \mathcal{S} \to \mathcal{L}$ be a homomorphism. We say that $(\phi, \mathcal{L})$ is the *universal semilattice* of $\mathcal{S}$ if every homomorphism from $\mathcal{S}$ to a semilattice factors through ϕ.

Fact 4.3.2 *Every semigroup has a universal semilattice.*

Fact 4.3.3 *For a finite monoid, the universal semilattice is a lattice.*

Fact 4.3.4 *Every finite lattice is the universal lattice of a finite commutative monoid (namely, of itself).*

Fact 4.3.5 *For a finite commutative monoid $\mathcal{M}$, the universal lattice is isomorphic to the lattice of idempotents of $\mathcal{M}$ and $\mathcal{M}$ is a lattice of semigroups with unique idempotent.*

Definition 4.3.6 Let $\mathcal{L} = \mathcal{L}(\mathcal{X}^*)$ denote the semilattice of idempotents of the sandpile monoid. We call $\mathcal{L}$ the *sandpile semilattice.*

Corollary 4.3.7 *$\mathcal{L}$ is the universal semilattice of $\mathcal{M}$.*

Theorem 4.3.8 *For a finite lattice $\mathcal{L}$, the following are equivalent:*

(i) $\mathcal{L}$ is the universal lattice of a sandpile monoid.

(ii) $\mathcal{L}$ is distributive.

The proof of Theorem 4.3.8 goes through a description of the sandpile semilattice $\mathcal{L}(\mathcal{X}^*)$ in terms of the strong components of the ambient space $\mathcal{X}^*$:

Theorem 4.3.9 *The following semilattices are isomorphic:*

(i) The sandpile semilattice $\mathcal{L}$.

(ii) The dual semilattice of normal ideals of the accessibility quasi-order $\mathcal{Q}(\mathcal{X}^*) = (V, \prec)$.

(iii) The dual semilattice of ideals of the accessibility preorder $\mathcal{P}(\mathcal{X}^*) = (V, \preccurlyeq)$.

(iv) The dual semilattice of ideals of the partial order $\mathcal{Q}(\mathcal{X}^*)^{ns} = (ns(\mathcal{X}^*), \preccurlyeq)$.

The equivalence of (ii), (iii) and (iv) is established in Proposition 2.1.10. To show the equivalence of (i) and (ii) we will need some more notation and lemmata.

Notation 4.3.10 Let $\mathbf{x} \in \mathcal{M}$ be a stable state. We denote by $\iota(\mathbf{x})$ the normal ideal generated by $\mathrm{supp}(\mathbf{x})$ in the accessibility quasi-order $\mathcal{Q}(\mathcal{X}^*) = (V, \prec)$ (see Notation 2.2.1).

Lemma 4.3.11 *Let the state $\mathbf{y}$ be reachable from the state $\mathbf{x}$. Then $\iota(\mathbf{x}) \subseteq \iota(\mathbf{y})$.*

Proof: It suffices to show that if C is an initial strong component of $\iota(\mathbf{x})$ then $C \subseteq \iota(\mathbf{y})$. If $C \cap \operatorname{supp}(\mathbf{x}) = \emptyset$, then C is reachable from an (abnormal) strong component C' of $I(\operatorname{supp}(\mathbf{x}))$. To obtain $\mathbf{y}$ from $\mathbf{x}$ we add grains to $\mathbf{x}$ and stabilize. If during this process we empty all abnormal strong components of $I(\operatorname{supp}(\mathbf{x}))$ from which C is reachable, then some grains have been transferred to C. By Lemma 4.1.6, we cannot empty the normal strong component C. If $C \cap \operatorname{supp}(\mathbf{x}) \neq \emptyset$, the result again follows from Lemma 4.1.6. □

Proposition 4.3.12 *Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{M}$. Then $\iota(\mathbf{x}_1 \oplus \mathbf{x}_2) = \iota(\mathbf{x}_1) \cup \iota(\mathbf{x}_2)$.*

Proof: We first show that $\iota(\mathbf{x}_1 \oplus \mathbf{x}_2) \subseteq \iota(\mathbf{x}_1) \cup \iota(\mathbf{x}_2)$. The state $\mathbf{x}_1 \oplus \mathbf{x}_2$ can have non-zero entries only on vertices accessible from $\operatorname{supp}(\mathbf{x}_1) \cup \operatorname{supp}(\mathbf{x}_2)$. So $\operatorname{supp}(\mathbf{x}_1 \oplus \mathbf{x}_2) \subseteq I(\operatorname{supp}(\mathbf{x}_1) \cup \operatorname{supp}(\mathbf{x}_2))$, and therefore $\iota(\operatorname{supp}(\mathbf{x}_1 \oplus \mathbf{x}_2)) \subseteq \iota(\operatorname{supp}(\mathbf{x}_1) \cup \operatorname{supp}(\mathbf{x}_2)) = \iota(\operatorname{supp}(\mathbf{x}_1)) \cup \iota(\operatorname{supp}(\mathbf{x}_2))$, and therefore, $\iota(\mathbf{x}_1 \oplus \mathbf{x}_2) \subseteq \iota(\mathbf{x}_1) \cup \iota(\mathbf{x}_2)$. The converse follows from Lemma 4.3.11.

□

Lemma 4.3.13 *Let $\mathbf{x} \in \mathcal{M}$. There exists $n > 0$ such that $n\mathbf{x}$ restricted to $\iota(\mathbf{x})$ is recurrent with respect to the ambient space $\mathcal{X}^*[\iota(\mathbf{x})]$, i. e., $n\mathbf{x}$ restricted to $\iota(\mathbf{x})$ belongs to $\mathcal{G}(\mathcal{X}^*[\iota(\mathbf{x})])$.*

Proof: We will denote the restriction of $\mathbf{x}$ on $\iota(\mathbf{x})$ by $\mathbf{x}'$. Let $C_1, C_2, \ldots, C_m$ be the initial strong components of $\iota(\mathbf{x})$ and let $u_s \in C_s$ for $s = 1, \ldots, m$. Every vertex in $\iota(\mathbf{x})$ is reachable from some u_s. By applying Lemma 4.1.9 for $u_s : s = 1, \ldots, m$ we find an $n > 0$ and a semistable state $\mathbf{h}$ such that $n\mathbf{x} = \sigma(\mathbf{h})$ and $h_v \geq \deg_*(v) - 1$ for all vertices $v \in \iota(\mathbf{x})$. So $\sigma(\mathbf{h})$ is reachable from the max state and the result follows. □

Lemma 4.3.14 *Let $\mathbf{e} \in \mathcal{L}$. Then $I(\operatorname{supp}(\mathbf{e}))$ is a normal ideal in $\mathcal{Q}(\mathcal{X}^*) = (V, \prec)$.*

Proof: Let C be an initial strong component of $I(\operatorname{supp}(\mathbf{e}))$. We will show that C is a normal strong component. Assume the opposite, i. e., assume that $C = \{v\}$

for some vertex v with no $v \to v$ loops. The idempotent satisfies $\mathbf{e} \oplus \mathbf{e} = \mathbf{e}$ and by assumption $\mathbf{e}(v) \neq 0$. If $2\mathbf{e}(v) < \deg_*(v)$, then clearly $(\mathbf{e} \oplus \mathbf{e})(v) \neq \mathbf{e}(v)$. If $2\mathbf{e}(v) \geq \deg_*(v)$, then $2\mathbf{e}(v) - \deg_*(v) > \mathbf{e}(v)$, again showing that $(\mathbf{e} \oplus \mathbf{e})(v) \neq \mathbf{e}(v)$. □

Lemma 4.3.15 *Let* $\mathbf{e} \in \mathcal{L}$. *Then* $\mathbf{e}$ *restricted to* $\iota(\mathbf{e})$ *is recurrent with respect to the ambient space* $\mathcal{X}^*[\iota(\mathbf{e})]$*, i. e.,* $\mathbf{e}$ *restricted to* $\iota(\mathbf{e})$ *belongs to* $\mathcal{G}(\mathcal{X}^*[\iota(\mathbf{e})])$.

Proof: By Lemma 4.3.14, $\mathrm{supp}(\mathbf{e}) \subseteq \iota(\mathbf{e})$. Now the result follows from Lemma 4.3.13 and the fact that $(\forall n > 0)(n\mathbf{e} = \mathbf{e})$. □

Corollary 4.3.16 *Let* $\mathbf{e} \in \mathcal{L}$. *Then* $\mathbf{e}$ *restricted to* $\iota(\mathbf{e})$ *is the identity in* $\mathcal{G}(\mathcal{X}^*[\iota(\mathbf{e})])$*, the sandpile group associated with* $\mathcal{X}^*[\iota(\mathbf{e})]$.

Proof: $\mathbf{e}$ is a recurrent idempotent in $\mathcal{M}(\mathcal{X}^*[\iota(\mathbf{e})])$ by Lemma 4.3.15. □

We are now in the position to show the following:

Proposition 4.3.17 *The map* ι *from* $\mathcal{L}$ *to the set of normal ideals of the accessibility quasi-order* $(V, \prec)$ *is 1-1 and onto.*

Proof: Let I be a normal ideal in the accessibility quasi-order in $\mathcal{X}^*$. Let $\mathbf{e}$ be the identity state in the ambient space $\mathcal{X}^*[I]$. Let C be an initial strong component of I. Since $\mathbf{e}$ is recurrent in the ambient space $\mathcal{X}^*[I]$, by Lemma 4.1.6, we have $\mathrm{supp}(\mathbf{e}) \cap C \neq \emptyset$. Therefore $\iota(\mathbf{e}) = I$. Now, let $I = \iota(\mathbf{e}_1) = \iota(\mathbf{e}_2)$. By Corollary 4.3.16, both $\mathbf{e}_1$ and $\mathbf{e}_2$ are equal to the identity state in the ambient space $\mathcal{X}^*[I]$, and therefore $\mathbf{e}_1 = \mathbf{e}_2$. □

Theorem 4.3.9 follows now from Propositions 4.3.12 and 4.3.17.

The characterization of ambient spaces for which there is a unique idempotent in the sandpile semigroup is an immediate consequence of Theorem 4.3.9.

Theorem 4.3.18 *The sandpile semigroup $\mathcal{S}$ has a unique idempotent if and only if either $\mathcal{X}$ is a DAG with at least one relevant vertex or $\mathcal{X}$ has a unique normal strong component.*

Proof: Immediate from Theorem 4.3.9. □

Corollary 4.3.19 *$\mathcal{L}$ is a distributive lattice.*

Proposition 4.3.20 *Every finite distributive lattice $\mathcal{L}$ is isomorphic to $\mathcal{L}(\mathcal{X}^*)$ for some ambient space $\mathcal{X}^*$.*

Proof: By a well-known reprentation theorem (see Theorem 15 [26], Chapter XIV, Section 6, p. 487), $\mathcal{L}$ is isomorphic to the semilattice of ideals of a partial order $\mathcal{P} = (V, \preccurlyeq)$. Define the graph $\mathcal{X}$ to have V as its vertex set and an edge from vertex v to u if $u \preccurlyeq v$. We obtain $\mathcal{X}^*$ by adding a sink vertex to $\mathcal{X}$ and having an out-edge from every vertex of V to the sink. □

This completes the proof of Theorem 4.3.8.

CHAPTER 5
BOUNDED NILPOTENCE CLASS

Assume the sandpile semigroup $\mathcal{S}$ has a unique idempotent, i. e., the ambient space $\mathcal{X}^*$ satisfies the conditions Theorem 4.3.18 and therefore the Rees quotient $\mathcal{S}/\mathcal{G}$ is nilpotent. Let k denote the nilpotence class of $\mathcal{S}/\mathcal{G}$. We observe that $k-1$ is the maximum weight of any (not necessarily stable) transient state $\mathbf{h} \in \mathbb{N}^V$.

5.1 Combinatorial characterization

Definition 5.1.1

(i) We define the *strong degree* of the vertex v (denoted by $\deg_s(v)$) to be the number of edges from v to the vertices in the strong component of v. So, an ordinary vertex v is abnormal if and only if $\deg_s(v) = 0$.

(ii) Let $\mathcal{X}^*$ be an ambient space and let A be the set of vertices that belong to all cycles in $\mathcal{X}$. We define the *effective volume* of $\mathcal{X}^*$ to be

$$\mathrm{vol}(\mathcal{X}^*) := \prod_{v \in A} \deg_s(v) \prod_{v \in V \setminus A} \deg_*(v). \tag{5.1}$$

Theorem 5.1.2 *For a class $\mathcal{C}$ of ambient spaces, the following are equivalent:*

(i) The nilpotence class of the Rees quotients $\mathcal{S}(\mathcal{X}^*)/\mathcal{G}(\mathcal{X}^*)$ is bounded ($\mathcal{X}^* \in \mathcal{C}$).

(ii) The number of transient states, $|\mathcal{M}(\mathcal{X}^*) \setminus \mathcal{G}(\mathcal{X}^*)|$, is bounded ($\mathcal{X}^* \in \mathcal{C}$).

(iii) The ambient spaces $\mathcal{X}^* \in \mathcal{C}$ have bounded effective volume.

In Section 5.2 we shall also prove that conditions (i), (ii), (iii) are equivalent to $\mathcal{X}^*$ having the explicit structure of a "circular tollway system" satisfying certain boundedness conditions. (Theorem 5.2.3).

Now, we focus on the proof of Theorem 5.1.2. The implication $(ii) \Rightarrow (i)$ is obvious. The implication $(iii) \Rightarrow (ii)$ is immediate from the following lemma which also motivates our definition of effective volume.

Lemma 5.1.3 *Let $\mathcal{X}^*$ be an ambient space. Then the number $|\mathcal{T}|$ of transient states satisfies $|\mathcal{T}| \leq \mathrm{vol}(\mathcal{X}^*)$.*

Proof: By Corollary 4.1.3, $|\mathcal{T}|$ is the number of functional subgraphs of $\mathcal{X}^*$ which contain a cycle. Let F be such a subgraph and let $v \in A$ and $v \to u$ be the unique out-edge of v in F. Since F contains a cycle C and $v \in C$, we have that $u \in C$ and therefore u belongs to the strong component of v. The edges going out of vertices in $V \setminus A$ can be directed anywhere. □

The rest of this section is devoted to showing the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ in Theorem 5.1.2.

Definition 5.1.4 By *contracting* an irrelevant edge $v \to u$ (see Definition 3.1.2) we mean deleting the vertex v and the edge $v \to u$ and replacing each edge of the form $\ell \to v$ by a new edge $\ell \to u$.

Note that contraction of an irrelevant edge $v \to u$ does not affect the relevance status of any vertex other than v so we can contract all irrelevant edges simultaneously; the resulting ambient space will have no irrelevant vertices (or edges).

Definition 5.1.5 Let $\mathcal{Y}$ be a digraph. The *relevant reduction* $\widetilde{\mathcal{Y}}$ of $\mathcal{Y}$ is the digraph obtained from $\mathcal{Y}$ by contracting all irrelevant edges.

A moment's reflection will verify the following observation.

Observation 5.1.6 *Let $\mathcal{X}^*$ be an ambient space.*

(i) We have $\mathcal{M}(\mathcal{X}^*) = \mathcal{M}(\widetilde{\mathcal{X}^*}), \mathcal{S}(\mathcal{X}^*) = \mathcal{S}(\widetilde{\mathcal{X}^*})$ and $\mathcal{G}(\mathcal{X}^*) = \mathcal{G}(\widetilde{\mathcal{X}^*})$.

(ii) $\mathrm{vol}(\mathcal{X}^*) = \mathrm{vol}(\widetilde{\mathcal{X}^*})$.

Observation 5.1.6 allows us to work in the relevant reduction $\widetilde{\mathcal{X}^*}$ of $\mathcal{X}^*$ instead of $\mathcal{X}^*$. Furthermore, we know from Corollary 4.2.7 that $\mathcal{S} = \mathcal{G}$ (so the sandpile quotient is trivial) if and only if $\mathcal{X}$ is a DAG with at least one relevant vertex.

Henceforth we shall thus make, without loss of generality, the following assumption:

Assumption: For all $v \in V$ we have $\deg_*(v) \geq 2$ and $\mathcal{X}$ contains at least one cycle. Morevover, the sink has outdegree 0. (For the last assumption we note that edges leaving the sink do not matter in the abelian sandpile model.)

Proposition 5.1.7 *Let C be a cycle in $\mathcal{X}$. Then*

$$|V \setminus V(C)| \leq k - 1 \quad \textit{and} \quad \sum_{v \notin V(C)} \deg_*(v) \leq 2(k-1).$$

Proof: Consider the stable state where all ordinary vertices outside C are saturated (every vertex $v \notin V(C)$ has $h_v = \deg_*(v) - 1$ grains) and C is blank ($h_v = 0$ for all $v \in V(C)$). By Proposition 4.1.6, this state $\mathbf{h}$ is transient. On the other hand it has weight $\sum_{v \notin V(C)} (\deg_*(v) - 1)$; therefore this sum is $\leq k-1$. The absence of irrelevant vertices implies that this sum is $\geq |V \setminus V(C)|$, proving that $|V \setminus V(C)| \leq k - 1$. Moreover, $\sum_{v \notin V(C)} \deg_*(v) = |V \setminus V(C)| + \sum_{v \notin V(C)} (\deg_*(v) - 1) \leq 2(k-1)$. □

Corollary 5.1.8 *If $\mathcal{X}$ has two vertex-disjoint cycles then $|E(\mathcal{X}^*)| \leq 4(k-1)$.*

Proof: Let C_1, C_2 be two vertex-disjoint cycles. Then $|E(\mathcal{X}^*)| = \sum_{v \in V} \deg_*(v) \leq \sum_{v \in V \setminus V(C_1)} \deg_*(v) + \sum_{v \in V \setminus V(C_2)} \deg_*(v) \leq 4(k-1)$. □

Henceforth we assume $|E(\mathcal{X}^*)| > 4(k-1)$. This implies that $\mathcal{X}$ has no two disjoint cycles. Let C be a shortest cycle of $\mathcal{X}$; let $\{v_0, \ldots, v_{n-1}\}$ denote the vertices of C in this cyclic order, so we have at least one edge $v_i \to v_{i+1}$ where addition in the subscript is modulo n. The minimality of n implies that:

Proposition 5.1.9 *For all i and s $(0 \le i, s \le n-1)$, any path from v_i to v_{i+s} has length $\ge s$.*

In particular, C is induced (there are no diagonals).

Let $P = (u_0 \to \cdots \to u_s)$ be a path in $\mathcal{X}$ such that $u_0 = v_i$ and $u_s = v_j$ but $u_1, \ldots, u_{s-1} \notin V(C)$. We call such a path a *bypass* of length s; the vertices bypassed are $v_{i+1}, \ldots, v_{j-1}$. By the preceding proposition, the number of vertices bypassed is $\le (s-1)$. We say that vertex $u \notin V(C)$ bypasses vertex v_i if there is a bypass through u that bypasses vertex v_i. Proposition 5.1.7 implies that vertex u cannot bypass more than $k-1$ vertices and that there are no more than $k-1$ vertices outside C that bypass vertices on C. so the total number of vertices of C which can be bypassed is $\le (k-1)^2$. Summarizing the foregoing, we obtain:

Proposition 5.1.10 *Let A denote the set of those vertices of $\mathcal{X}$ which belong to all cycles and B the set of those vertices of C which are bypassed. Then $V(C) = A \dot\cup B$ and $|B| \le (k-1)^2$.*

Corollary 5.1.11 *Let A be the set of vertices of $\mathcal{X}$ which belong to all cycles. Then $|V \setminus A| \le k(k-1)$.*

Proof: Using the notation of the preceding proposition, it follows from Proposition 5.1.7 that $|V \setminus V(C)| \le k-1$. Moreover, $|V(C) \setminus A| \le (k-1)^2$ by Proposition 5.1.10. Adding these two inequalities gives the desired bound.

□

We also note that $\deg_*(u) \le 2(k-1)$ for all $u \notin A$ by Proposition 5.1.7; therefore

$$|E(\mathcal{X}^*)| - \sum_{u \notin A} \deg_*(u) \le 2k^3. \tag{5.2}$$

(We note that this bound can be improved to $O(k)$.) So for fixed k, all but a bounded number of edges originate from vertices that belong to all cycles.

Recall that $\mathcal{X}^*$ has only one normal strong component (because of the uniqueness of the idempotent and the assumption that $\mathcal{X}^*$ is not a DAG).

We now turn to the description of this normal strong component, K.

Recall that the strong degree $\deg_s(v)$ of the vertex v is the number of edges from v to the vertices in the strong component of v.

Definition 5.1.12 Let $\mathcal{Y}$ be a digraph.

(i) We say that the vertex v is *thin* if $\deg_s(v) = 1$; a vertex v with $\deg_s(v) \geq 2$ is *fat*.

(ii) We say that the edge $v \to u$ is *thin* if the vertex v is thin.

(iii) A subgraph of $\mathcal{Y}$ is *thin* if all its edges are thin in $\mathcal{Y}$.

Note that by our assumption of the absence of irrelevant vertices, each thin vertex is the tail of at least one edge directed outside K (since $\deg_*(u) \geq 2$).

Recall that in accordance with Definition 5.1.12, a *thin path* is a path $(u_0 \to \cdots \to u_k)$ where $\deg_s(u_0) = \cdots = \deg_s(u_{k-1}) = 1$.

Recall that the state $\mathbf{t}_v$ is defined by putting one grain on v and none elsewhere.

Definition 5.1.13 We call the vertex $v \in V$ *happy* if the state $\mathbf{t}_v$ is recurrent.

Lemma 5.1.14 *Assume $(u_0 \to \cdots \to u_k)$ is a thin path in K. Then u_k is happy.*

Proof: Consider the states $\mathbf{w}$ defined by saturating each vertex u_i for $0 \leq i \leq k-1$. This state is recurrent because its weight is $\geq k$. Now add one grain at u_0; call the state obtained $\mathbf{w'}$. This state is unstable; stabilization will occur by toppling u_0, then u_1, etc., u_{k-1}. Each toppling passes one grain to the next vertex; so the final stable state will be $\mathbf{w} = \sigma(\mathbf{t}_{u_k} + \mathbf{h})$, where $\mathbf{h}$ is a state concentrated on vertices outside of and accessible from K. Noting that all these vertices are points of no return, we infer from Lemma 4.2.4 that 0 is accessible from $\sigma(\mathbf{h})$ and therefore $\mathbf{t}_{u_k}$ is accessible from $\mathbf{w}$. Hence, $\mathbf{t}_{u_k}$ is recurrent. □

Let H denote the set of happy vertices, T the set of thin vertices, and F the set of fat vertices. Recall that A denotes the set of those vertices which belong to all cycles.

Lemma 5.1.15 $|A \setminus (F \cup H)| \leq k|F|$.

Proof: Note that the subgraph induced by $A \setminus F$ consists of disjoint thin paths and has $\leq |F|$ components. According to Lemma 5.1.14, all but k of the vertices of each segment are happy. □

Corollary 5.1.16

$$|V \setminus H| \leq (k+1)|F| + k^2.$$

Proof: $V \setminus H \subseteq F \cup (A \setminus (F \cup H)) \cup (V(C) \setminus A) \cup (V \setminus V(C)$. Therefore, by Lemma 5.1.15 and Propositions 5.1.10 and 5.1.7, we obtain that $|V \setminus H| \leq |F| + k|F| + (k-1)^2 + (k-1)$. □

We are now ready to make our key inference that all but a bounded number of vertices in K are thin. We first prove a lemma.

Lemma 5.1.17 $|\mathcal{T}| \leq (\mathrm{e}r)^{k-1}$ *where* $r = |V \setminus H|$.

Proof: No happy vertex can be part of the support of a transient state, and no vertex can have height $\geq k$ in a transient state, so the number of transient states is less than $\binom{r}{k-1} k^{k-1} < (\mathrm{e}r)^{k-1}$. □

Theorem 5.1.18 $|F| \leq 4k \log k + 4$. *("log" refers to base-2 logarithms.)*

Proof: By Corollary 4.1.4, the number of transient states is $|\mathcal{T}| \geq 2^{|F|}$.

By Corollary 5.1.16, we have $r \leq (k+1)|F| + k^2$. Combining all inequalities we obtain that

$$2^{|F|} < \left(\mathrm{e}(k+1)|F| + k^2\right)^{k-1}.$$

For fixed k, the left-hand side grows exponentially while the right-hand side polynomially in $|F|$ so $|F|$ is bounded as a function of k. Some calculation gives the bound stated. □

It follows that all but a bounded number of vertices are both happy and thin.

Corollary 5.1.19

$$|(V \setminus H) \cup F| \leq 4(k+2)^2(1+\log k).$$

Proof: Combine Theorem 5.1.18 with Corollary 5.1.16. □

Our main corollary now follows: the number of transient states is bounded as a function of k.

Corollary 5.1.20

$$|\mathcal{T}| < \max\{C, k^{3k}\},$$

where C is an absolute constant.

Proof: By Lemma 5.1.17, $|\mathcal{T}| < (\mathrm{e}r)^{k-1}$, where $r = |V \setminus H|$ is the number of unhappy vertices. Now use the estimate on r from the preceding corollary. □

This completes the proof of the $(i) \Rightarrow (ii)$ implication of Theorem 5.1.2 and thereby the proof of the first part of Theorem 1.4.1.

We now move on to proving the implication $(ii) \Rightarrow (iii)$. Most of what we need has already been inferred from (i); we now draw further conclusions from (ii).

It follows that relative to $\mathcal{X}$, all vertices have bounded degree. Indeed, since $|\mathcal{T}| \geq \prod_{u \in V} \deg(u)$, a very weak conclusion is that

Corollary 5.1.21 *For all $u \in V$, we have* $\deg(u) \leq |\mathcal{T}| < \max\{C, k^{3k}\}$.

We expect that a much stronger upper bound, perhaps $\deg(u) = O(k)$, holds.

Now we need to prove that the effective volume (Definition 5.1.1) of $\mathcal{X}^*$ is bounded.

By Proposition 5.1.7 we have $\deg_*(v) \leq 2(k-1)$ for all $v \in V \setminus A$. By Corollary 5.1.11 we have $|V \setminus A| \leq k(k-1)$. Combining these two inequalities we obtain

$$\prod_{v \in V \setminus A} \deg_*(v) \leq (2(k-1))^{k(k-1)}. \tag{5.3}$$

By Corollary 4.1.4,

$$\prod_{v\in A} \deg_s(v) \leq \prod_{v\in V} \deg(v) \leq |\mathcal{T}|. \tag{5.4}$$

A combination of equations (5.3) and (5.4) yields that the effective volume of $\mathcal{X}^*$ is bounded. This completes the proof of the implication $(ii) \Rightarrow (iii)$ of Theorem 5.1.2.

5.2 Explicit structure

In this section we define the structure of a "circular tollway system" and we state our theorem on the asymptotic characterization of the ambient spaces for which the sandpile quotient has bounded nilpotence class (Theorem 5.2.3).

Definition 5.2.1 Let $\mathcal{Y}$ be a directed acyclic graph (DAG) and let v be a vertex.

(i) The vertex v is an *entrance* if every vertex of $\mathcal{Y}$ is accessible from v.

(ii) The vertex v is an *exit* if v is accessible from every vertex of $\mathcal{Y}$.

(iii) $\mathcal{Y}$ is a *rest area* if it has an entrance and an exit. Note that the entrance and the exit are unique.

(iv) The *interior* vertices of a rest area are the vertices other than the entrance and the exit. We denote the set of interior vertices of the rest area $\mathcal{Y}$ by $\text{int}(\mathcal{Y})$.

Definition 5.2.2

(i) A *circular highway* is a strong component of $\mathcal{X}$ which is a thin directed cycle.

(ii) We now construct a *circular tollway.* We start with a circular highway on which we designate edges $u_1 \to v_1, u_2 \to v_2, \ldots, u_n \to v_n$ in this cyclic order ($n \geq 0$). We take rest areas $R_1, \ldots, R_n$ (the R_i are disjoint from each other and from the the circular highway). For $i = 1, \ldots, n$, we delete the edge $u_i \to v_i$ (but we keep the vertices) and we glue a rest area R_i between u_i and v_i so that $\text{entrance}(R_i) = u_i$ and $\text{exit}(R_i) = v_i$.

(iii) We say that an ambient space $\mathcal{X}^*$ is a *circular tollway system* if $\mathcal{X}$ has a unique normal strong component and this strong component is a circular tollway.

Theorem 5.2.3 *For a class $\mathcal{C}$ of ambient spaces, the following are equivalent:*

(i) The nilpotence class of the Rees quotients $\mathcal{S}(\mathcal{X}^*)/\mathcal{G}(\mathcal{X}^*)$ is bounded ($\mathcal{X}^* \in \mathcal{C}$).

(ii) The number of transient states $|\mathcal{M}(\mathcal{X}^*)\backslash\mathcal{G}(\mathcal{X}^*)|$ is bounded ($\mathcal{X}^* \in \mathcal{C}$).

(iii) The effective volume of $\mathcal{X}^*$ is bounded.

(iv) $(\exists n_0 \geq 0)$ (if $\mathcal{X}^* \in \mathcal{C}$ has more than n_0 relevant edges then $\mathcal{X}^*$ is a circular tollway system of bounded effective volume).

By Theorem 5.1.2, *(i)*, *(ii)* and *(iii)* are equivalent. *(iv)* $\Rightarrow$ *(iii)* is obvious. To complete the proof of Theorem 5.2.3 we need to show that *(iii)* $\Rightarrow$ *(iv)*. The proof is easy and omitted.

5.3 Structure of the sandpile group

We shall use the explicit structure of our ambient spaces to prove that the sandpile group is cyclic-by-bounded, thereby proving the second part of Theorem 1.4.1.

First we prove two general statements.

Lemma 5.3.1 *Assume $\mathcal{X}$ has s initial strong components. Assume all ordinary vertices have* $\deg \leq d$ *and all but r of the ordinary vertices have* $\deg = 1$. *Then the sandpile monoid $\mathcal{M}$ is generated by $\leq rd + s$ elements.*

Proof: Let $B = \{u \in V : \deg(u) \geq 2\}$. Let N denote the set of ordinary (non-root) out-neighbors of vertices in B. Clearly, $|N| \leq rd$. Let S be a set of representatives of the initial strong components of $\mathcal{X}$ (one vertex per initial strong component). We claim that the set $T = \{\mathbf{t}_u : u \in N \cup S\}$ generates $\mathcal{S}$.

Note that every vertex of $\mathcal{X}$ is accessible from S. Let $j \in V$ and let ℓ be the directed distance from $N \cup S$ to j. We need to show that $\mathbf{t}_j$ is generated by T. We prove this by induction on ℓ. If $\ell = 0$ then $j \in N \cup S$ and we are done. Suppose

$\ell \geq 1$ and let $u \to j$ be the last step of a shortest path from $N \cup S$ to j. Then, by the inductive hypothesis, $\mathbf{t}_u$ is generated by T. Also, $\deg(u) = 1$ (otherwise we would have $j \in N$). Consequently, $\mathbf{t}_j = \mathbf{t}_u^{\deg_*(u)}$. □

Corollary 5.3.2 *Assume $\mathcal{X}$ has a unique strong component K. Let D be the set of ordinary vertices accessible from but not belonging to K. Let $\mathcal{X}$ have s initial strong components. Assume all ordinary vertices have* $\deg \leq d$ *and all but r of the vertices are thin. Then the sandpile group $\mathcal{G}$ is generated by $\leq rd+s+|D|$ elements. Moreover, if all vertices in D have* $\deg_* \leq d^*$*, then $\mathcal{G}$ has a subgroup $\mathcal{D}$ of order $\leq (d^*)^{|D|}$ such that $\mathcal{G}/\mathcal{D}$ is the sandpile group of the contraction $\mathcal{X}^*/D$.*

Proof: Let $\mathcal{G}' = \mathcal{G}/\mathcal{D}$ where $\mathcal{D}$ is the subgroup generated by the set $\{x_v : v \in D\}$. Then, by Corollary 3.2.13, $\mathcal{G}' = \mathcal{G}(\mathcal{X}^*/D)$. Let $f : V^* \to V^* \setminus D$ be the contraction map. If v is a thin vertex in $\mathcal{X}^*$, then $\deg(f(v)) = 1$ in $\mathcal{X}^*/D$. Therefore, an application of Lemma 5.3.1 yields that the rank of $\mathcal{G}'$ is at most $rd + s$ and therefore the rank of $\mathcal{G}$ is at most $rd + s + |D|$. Moreover, $|\mathcal{D}| \leq (d^*)^{|D|}$. □

We need the full power of the "circular tollway of bounded effective volume" structure to deduce our conclusion that $\mathcal{G}$ has a cyclic subgroup of bounded index. We sketch the proof.

First, by the last sentence of Corollary 5.3.2 we may assume that the only vertex accessible from but not belonging to K is the sink. (Otherwise we contract the set D to the sink; the effect is that $\mathcal{G}$ is reduced to $\mathcal{G}/\mathcal{D}$ where $\mathcal{D}$ has bounded order.)

Let the rest areas be $R_0, \ldots, R_{n-1}$ in this cyclic order and let $v_i := \text{exit}(R_i)$. So, v_i is the start of the thin path leading to R_{i+1}. Clearly, $\mathbf{t}_{v_i}$ generates $\mathbf{t}_v$ for all v along this path. Taking further multiples of $\mathbf{t}_{v_i}$ "invades" the rest area R_{i+1} (addition in subscripts modulo n); but R_{i+1} is a DAG, so we can "flush it" by Lemma 4.2.4; we then obtain a bounded multiple of $\mathbf{t}_{v_{i+1}}$. Repeating this around the cycle we find that a bounded multiple of every generator belongs to the cyclic group $\mathcal{H}$ generated by $\mathbf{t}_{v_1}$. So, $\mathcal{G}/\mathcal{H}$ has bounded exponent. But $\mathcal{G}$, and therefore

$\mathcal{G}/\mathcal{H}$ also has bounded rank by Lemma 5.3.1; hence $\mathcal{G}/\mathcal{H}$ has bounded order. $\square$

CHAPTER 6
SANDPILE GROUP FOR COMPLETE REGULAR TREES

6.1 Lattices: preliminaries

Notation 6.1.1 Let G be a group and let $\mathbf{z} \in G$. We denote the order of $\mathbf{z}$ in G by $\mathrm{ord}(\mathbf{z})$.

The following lemma will be used repeatedly:

Lemma 6.1.2 *Let $\mathbf{u}_1, \ldots, \mathbf{u}_t \in \mathbb{Z}^t$ be linearly independent over $\mathbb{Q}$ and let $\Lambda := \sum_{n=1}^{t} \mathbb{Z}\mathbf{u}_n$ be the lattice spanned by the $\mathbf{u}_n$. Consider the finite abelian group $K = \mathbb{Z}^t/\Lambda$. Assume $\mathbf{u} \in \mathbb{Z}^t$ satisfies*

$$r\mathbf{u} = \sum_{n=1}^{t} r_n \mathbf{u}_n, \tag{6.1}$$

where $r, r_n \in \mathbb{Z}$, $r > 0$ and $\gcd(r_1, \ldots, r_n) = 1$. Then the order of $\mathbf{u}$ in K is r.

Proof: Let $\mathbf{u} \mapsto \bar{\mathbf{u}}$ denote the quotient map $\mathbb{Z}^t \to \mathbb{Z}^t/\Lambda$. By definition we have $r\mathbf{u} \in \Lambda$, therefore $\mathrm{ord}(\bar{\mathbf{u}}) \mid r$. Let $r = \mathrm{ord}(\bar{\mathbf{u}})s$. The relation $\mathrm{ord}(\bar{\mathbf{u}})\mathbf{u} \in \Lambda$ implies

$$\mathrm{ord}(\bar{\mathbf{u}})\mathbf{u} = \sum_{n=1}^{t} q_n \mathbf{u}_n, \tag{6.2}$$

where $q_n \in \mathbb{Z}$. Multiplying (6.2) by s we obtain

$$r\mathbf{u} = \sum_{n=1}^{t} s q_n \mathbf{u}_n. \tag{6.3}$$

The $\mathbf{u}_n$ are linearly independent and thus equations (6.1) and (6.3) yield $r_n = sq_n$ $(1 \leq n \leq t)$. So $s \mid \gcd(r_1, \ldots, r_n)$ and therefore $s = 1$. □

Definition 6.1.3 Let s be a positive integer and let p be a prime.

(i) We define $e_p(s) := w$ if $p^w \mid s$ and $p^{w+1} \nmid s$.

(ii) Let G be group and $\mathbf{z} \in G$. We define $e_p(\mathbf{z}) := e_p(\mathrm{ord}(\mathbf{z}))$.

The following Lemma will be used in the proof of Theorem 1.5.7.

Lemma 6.1.4 *Let G be an abelian group and let $\mathbf{z}_n \in G$ $(0 \leq n \leq t)$. Let $\mathrm{ord}(\mathbf{z}_0) = s$. Assume $r_n\mathbf{z}_{n+1} = r_{n+1}\mathbf{z}_n$ $(0 \leq n \leq t-1)$, where $r_n \in \mathbb{Z}$ and $\gcd(r_n, r_{n+1}) = 1$ $(0 \leq n \leq t-1)$. Then $\mathrm{ord}(\mathbf{z}_n) \mid \mathrm{lcm}\{sr_0, r_1, \ldots, r_{n-1}\}$ $(1 \leq n \leq t)$.*

Proof: Let p be a prime.

Claim 6.1.5

$$e_p(\mathbf{z}_n) \leq \max\{e_p(\mathbf{z}_{n-2}), e_p(\mathbf{z}_{n-1}), e_p(r_{n-1})\} \ (2 \leq n \leq t). \tag{6.4}$$

Proof of Claim 6.1.5: We have

$$r_{n-1}\mathbf{z}_n = r_n\mathbf{z}_{n-1}. \tag{6.5}$$

If $p \nmid r_{n-1}$ then by equation (6.5) we have $e_p(\mathbf{z}_n) \leq e_p(\mathbf{z}_{n-1})$.

If $p \mid r_{n-1}$ consider the equation

$$r_{n-2}\mathbf{z}_{n-1} = r_{n-1}\mathbf{z}_{n-2}. \tag{6.6}$$

We have $p \nmid r_{n-2}r_n$.

If $e_p(r_{n-1}) < e_p(\mathbf{z}_{n-2})$, then by equation (6.6) we obtain

$$e_p(\mathbf{z}_{n-1}) = e_p(\mathbf{z}_{n-2}) - e_p(r_{n-1}) > 0. \tag{6.7}$$

So by equation (6.5) we obtain $e_p(\mathbf{z}_n) = e_p(\mathbf{z}_{n-1}) + e_p(r_{n-1})$. By (6.7) this is $e_p(\mathbf{z}_{n-2})$.

If $e_p(r_{n-1}) \geq e_p(\mathbf{z}_{n-2})$ then by equation (6.6) we obtain $e_p(\mathbf{z}_{n-1}) = 0$ and by equation (6.5) we obtain $e_p(\mathbf{z}_n) \leq e_p(r_{n-1})$. □

For the proof of the Lemma, it suffices to prove the following:

$$e_p(\mathbf{z}_n) \leq \max\{e_p(s) + e_p(r_0), e_p(r_1), \dots, e_p(r_{n-1})\} \ (1 \leq n \leq t). \tag{6.8}$$

We prove (6.8) by induction on n:

Let $n = 1$. We have

$$r_0\mathbf{z}_1 = r_1\mathbf{z}_0$$

and therefore

$$r_0 s\mathbf{z}_1 = r_1 s\mathbf{z}_0 = 0.$$

Let $n = 2$. By the Claim we have

$$e_p(\mathbf{z}_2) \leq \max\{e_p(\mathbf{z}_0), e_p(\mathbf{z}_1), e_p(r_1)\} \leq \max\{e_p(s) + e_p(r_0), e_p(r_1)\}.$$

For $n \geq 3$, the inductive step is accomplished by Claim 6.1.5. □

6.2 Trees: preliminaries and notation

From now on $\mathcal{T} = \mathcal{T}(d) = (V(\mathcal{T}(d)), E(\mathcal{T}(d)))$ denotes the infinite d-valent tree. The distance of the vertices i and j is denoted by $\mathrm{dist}(i,j)$. We designate a vertex to be the root and we denote it by 0. Let $i, j \in V(\mathcal{T}(d))$. We say that i is an *ancestor* of j or j is a *descendant* of i ($i \preceq j$), if i is on the unique path from j to 0 and that j is a *proper descendant* of i ($i \prec j$), if $i \preceq j$ and $i \neq j$. Let $V_i = \{k : i \preceq k\}$ be the set of descendants of i. We say that i is the *parent* of j ($i = p(j)$) or that j is a *child* of i, if $i \prec j$ and $\mathrm{dist}(i,j) = 1$. Let $C_i = \{k : p(k) = i\}$ be the set of children of i. In particular, let $C_0 = \{1, \dots, d\}$ denote the set of children of the root 0. We say that i and j are *siblings* if $p(i) = p(j)$ and $i \neq j$. Let B_n be the ball of radius n about the root, i. e., $B_n := \{k \in V(\mathcal{T}(d)) : \mathrm{dist}(0,k) \leq n\}$ and let S_n be the sphere of radius n about the root, i. e., $S_n := \{k \in V(\mathcal{T}(d)) : \mathrm{dist}(0,k) = n\}$. Let $V_i^n := V_i \cap S_n$ be the set of descendants of i in depth n.

We define the complete d-valent tree of depth h, denoted by $\mathcal{T}(d,h) = (V(\mathcal{T}(d,h)), E(\mathcal{T}(d,h)))$ as the subgraph of $\mathcal{T}(d)$ induced on B_h. A vertex i is called a *leaf* if $\mathrm{dist}(0,i) = h$. In our notation, $V_i^h = V_i \cap S_h$ is the set of leaves below the vertex i. Let $\hat{\mathcal{T}}(d,h)$ be the graph obtained from $\mathcal{T}(d,h)$ by attaching an additional vertex (the sink) by $d-1$ edges to each leaf. We denote the sandpile group of $\hat{\mathcal{T}}(d,h)$ by $G(d,h)$ and call it the *sandpile group of the d-valent tree of depth h.* The order of an element $\mathbf{x}$ in $G(d,h)$ is denoted by $\mathrm{ord}(\mathbf{x})$.

In the case of $G(d,h)$, the lattice Λ is generated by the following vectors ($\mathbf{x}_i$ stands for the generator of $G(d,h)$ corresponding to the vertex i).

$$\boldsymbol{\delta}_i := \begin{cases} d\mathbf{x}_i - \mathbf{x}_{p(i)} - \sum_{j \in C_i} \mathbf{x}_j & \text{if } \mathrm{dist}(0,i) < h, \\ d\mathbf{x}_i - \mathbf{x}_{p(i)} & \text{otherwise.} \end{cases} \tag{6.9}$$

The numbers $\theta(d,n)$ defined as follows occur many times throughout the analysis:

$$\theta(d,n) := \frac{(d-1)^n - 1}{d-2}. \tag{6.10}$$

The $\theta(d,n)$ satisfy the recurrence

$$\theta(d,n+2) = d\theta(d,n+1) - (d-1)\theta(d,n). \tag{6.11}$$

6.3 The rank

In this section we prove that the rank of $G(d,h)$ is $(d-1)^h$ (Theorem 1.5.1).

First, we introduce some notation and define a subset $F(d,h)$ of the vertex set of $\mathcal{T}(d,h)$, with $|F(d,h)| = (d-1)^h$. We shall see that the generators $\{\bar{\mathbf{x}}_i : i \in F(d,h)\}$ generate $G(d,h)$ (Lemma 6.3.6). Recall that we denote the root by 0 and that C_i is the set of children of the vertex i.

Definition 6.3.1 Let $i \in V(\mathcal{T}(d,h))$. Let us pick a child m_i of i. Set $J_i := C_i \backslash \{m_i\}$.

Definition 6.3.2 If h is even, then let

$$F(d,h) := \{0\} \cup \left(\bigcup_{q=0}^{(h-2)/2} \bigcup_{i \in S_{2q+1}} J_i \right).$$

If h is odd, then let

$$F(d,h) := \bigcup_{q=0}^{(h-1)/2} \bigcup_{i \in S_{2q}} J_i.$$

Note 6.3.3

(i) If h even, then $|F(d,h)| = 1 + (d-2)\sum_{q=0}^{(h-2)/2} d(d-1)^{2q} =$
$= 1 + d(d-2)[(d-1)^h - 1]/[(d-1)^2 - 1] = (d-1)^h$.
If h odd, then $|F(d,h)| = (d-1) + (d-2)\sum_{q=1}^{(h-1)/2} d(d-1)^{2q-1} =$
$= (d-1) + (d-2)d(d-1)\sum_{q=0}^{(h-3)/2}(d-1)^{2q} =$
$= (d-1) + (d-2)d(d-1)[(d-1)^{h-1} - 1]/[(d-1)^2 - 1] =$
$= (d-1) + (d-1)[(d-1)^{h-1} - 1] = (d-1)^h$.

(ii) Observe that $F(d,h+2) = F(d,h) \cup (\bigcup_{i \in S_{h+1}} J_i)$.

Notation 6.3.4

(i) Let $\Lambda := \sum_{i \in V(\mathcal{T}(d,h))} \mathbb{Z}\boldsymbol{\delta}_i$ denote the lattice generated by the δ_i.

(ii) Let ϕ denote the quotient map from $\mathbb{Z}^{V(\mathcal{T}(d,h))}$ to $G(d,h) := \mathbb{Z}^{V(\mathcal{T}(d,h))}/\Lambda$.

(iii) Let $\mathbf{v} \in \mathbb{Z}^{V(\mathcal{T}(d,h))}$. We denote $\phi(\mathbf{v})$ by $\bar{\mathbf{v}}$.

Definition 6.3.5 Let $G_F(d,h)$ be the subgroup of $G(d,h)$ generated by $\{\bar{\mathbf{x}}_i : i \in F(d,h)\}$.

The proof of Theorem 1.5.1 is based on the next two lemmas:

Lemma 6.3.6 $G_F(d,h) = G(d,h)$.

Proof: In this proof we will work in $G(d,h)/G_F(d,h)$, so, e.g., the statement "$\bar{\mathbf{x}}_i = \bar{\mathbf{x}}_j$" means "$\bar{\mathbf{x}}_i + G_F(d,h) = \bar{\mathbf{x}}_j + G_F(d,h)$." Note that under this convention, $\bar{\mathbf{x}}_i = 0$ for all $i \in F(d,h)$. Also, note that by the definition of $G(d,h)$, $\bar{\boldsymbol{\delta}}_i = 0$ for all $i \in V(\mathcal{T}(d,h))$.

We prove by induction on h that $G(d,h)/G_F(d,h) = 0$. We will show this by showing that $(\forall i \in V(\mathcal{T}(d,h)))(\bar{\mathbf{x}}_i = 0)$.

If $h = 0$, this is obvious.

If $h = 1$, then by the definition of $F(d,1)$ we have $(\forall i \in J_0)(\bar{\mathbf{x}}_i = 0)$. Also, $(\forall i \in J_0)(\bar{\boldsymbol{\delta}}_i = 0)$, so $(\forall i \in J_0)(d\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_0 = 0)$, and therefore $\bar{\mathbf{x}}_0 = 0$. Finally, $\bar{\boldsymbol{\delta}}_0 = 0$, so $d\bar{\mathbf{x}}_0 - \sum_{i \in J_0} \bar{\mathbf{x}}_i - \bar{\mathbf{x}}_{m_0} = 0$, and thus $\bar{\mathbf{x}}_{m_0} = 0$.

Assume that the statement is true for h and consider the case $h+2$: We have $\bigcup_{i \in S_{h+1}} J_i \subseteq F(d,h+2)$, so $(\forall i \in S_{h+1})(\forall k \in J_i)(\bar{\mathbf{x}}_k = 0)$. Also, $(\forall i \in S_{h+1})(\forall k \in J_i)(\bar{\boldsymbol{\delta}}_k = 0)$, so $(\forall i \in S_{h+1})(\forall k \in J_i)(d\bar{\mathbf{x}}_k - \bar{\mathbf{x}}_i = 0)$, and hence $(\forall i \in S_{h+1})(\bar{\mathbf{x}}_i = 0)$. So, all relations of the form $\bar{\boldsymbol{\delta}}_i = 0$ with $i \in B_h$ for $G(d,h+2)/G_F(d,h+2)$ become identical with the corresponding ones for $G(d,h)/G_F(d,h)$, and therefore by the inductive assumption we obtain $(\forall i \in B_h)(\bar{\mathbf{x}}_i = 0)$. Finally, for $i \in S_{h+1}$ we have $\bar{\boldsymbol{\delta}}_i = 0$, therefore $d\bar{\mathbf{x}}_i - \bar{\mathbf{x}}_{p(i)} - \sum_{k \in J_i} \bar{\mathbf{x}}_k - \bar{\mathbf{x}}_{m_i} = 0$, and thus $\bar{\mathbf{x}}_{m_i} = 0$. □

Corollary 6.3.7 *The* rank *of the sandpile group $G(d,h)$ is $\leq (d-1)^h$.*

Lemma 6.3.8 *$G(d,h)$ contains a copy of $\underbrace{\mathbb{Z}_d \oplus \cdots \oplus \mathbb{Z}_d}_{(d-1)^h \text{ times}}$.*

Proof: We work in $\pmod{d}$ arithmetic. For $i \in F(d,h)$, fix $r_i \in \mathbb{Z}$ and consider the system

$$\begin{cases} \Delta \mathbf{v}^T &= 0, \quad \mathbf{v} = (v_i)_{i \in V(\mathcal{T}(d,h))}, \\ v_i &= r_i, \quad i \in F(d,h). \end{cases} \qquad (6.12)$$

System (6.12) is equivalent to

$$\begin{cases} \boldsymbol{\delta}_i \cdot \mathbf{v} &= 0, \quad i \in V(\mathcal{T}(d,h)), \\ \quad v_i &= r_i, \quad i \in F(d,h), \end{cases} \qquad (6.13)$$

where $\mathbf{u} \cdot \mathbf{v}$ denotes the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ (mod d).

Claim 6.3.9 *For any choice of $r_i \in \mathbb{Z}$, system (6.13) has a solution in $\mathbb{Z}^{V(\mathcal{T}(d,h))}$.*

Proof of Claim 6.3.9: By induction on h.

If $h = 0$, this is clear.

If $h = 1$, we have $F(d,1) = J_0$. Set $s_{1,0} := 0$, $s_{1,m_0} := -\sum_{i \in J_0} r_i$ and $s_{1,i} := r_i$ for $i \in J_0$. Then $\mathbf{s}_1 = (s_{1,i})_{i \in V(\mathcal{T}(d,1))}$ is a solution of (6.13).

Consider (6.13) for the tree of depth $h+2$ with chosen constants r_i for $i \in F(d, h+2)$. Let $\mathbf{s}_h = (s_{h,i})_{i \in V(\mathcal{T}(d,h))}$ be a solution of (6.13) for the h case with constants r_i for $i \in F(d,h)$. Set $s_{h+2,i} := s_{h,i}$ for all $i \in B_h$, $s_{h+2,i} := 0$ for $i \in S_{h+1}$, $s_{h+2,m_i} = -\sum_{k \in J_i} r_k - s_{h,p(i)}$ for all $i \in S_{h+1}$ and $s_{h+2,i} := r_i$ for all $i \in F(d, h+2)$. Then $\mathbf{s}_{h+2} := (s_{h+2,i})_{i \in V(\mathcal{T}(d,h+2))}$ is a solution of (6.13). $\square$

Claim 6.3.9 yields that for all $k \in F(d,h)$ there exists a solution vector $\mathbf{v}_k = (v_{k,i})_{i \in V(\mathcal{T}(d,h))}$ of (6.13) such that $v_{k,k} = 1$, and for all $i \in F(d,h)$ with $i \neq k$ we have $v_{k,i} = 0$.

For $\mathbf{v} \in \mathbb{Z}^{V(\mathcal{T}(d,h))}$, let $\tilde{\mathbf{v}} := (v_i)_{i \in F(d,h)}$. Note that $\{\tilde{x}_i : i \in F(d,h)\}$ is a basis of $\mathbb{Z}_d^{F(d,h)}$.

Consider the homomorphism $\omega : \mathbb{Z}^{V(\mathcal{T}(d,h))} \to {\mathbb{Z}_d}^{F(d,h)}$ with

$$\omega(\mathbf{u}) := (\mathbf{u} \cdot \mathbf{v}_k)_{k \in F(d,h)}. \qquad (6.14)$$

For all $k \in F(d,h)$ we have $\omega(\mathbf{x}_k) = \tilde{\mathbf{x}}_k$. Therefore ω is surjective. Also, for all $k \in F(d,h)$, $\mathbf{v}_k$ is a solution of (6.13). Therefore $(\forall k \in F(d,h))(\forall i \in V(\mathcal{T}(d,h)))(\boldsymbol{\delta}_i \cdot \mathbf{v}_k = 0)$. So, $(\forall i \in V(\mathcal{T}(d,h)))(\omega(\boldsymbol{\delta}_i) = 0)$. Therefore there exists a homomorphism $\psi : G \to {\mathbb{Z}_d}^{F(d,h)}$ with $\omega = \psi \circ \phi$. The map ω is surjective therefore ψ is surjective. $\square$

6.4 The exponent

In this section we show that the exponent $\exp(d,h)$ of $G(d,h)$ is equal to $(d-1)^h \cdot \operatorname{lcm}\{d\theta(d,h+1), \theta(d,h), \theta(d,h-1), \ldots, \theta(d,2)\}$ (Theorem 1.5.7).

Recall that $\Lambda := \sum_{i \in V(\mathcal{T}(d,h))} \mathbb{Z}\boldsymbol{\delta}_i$ is the lattice spanned by the δ_i, that ϕ denotes the quotient map from $\mathbb{Z}^{V(\mathcal{T}(d,h))}$ to $G(d,h) := \mathbb{Z}^{V(\mathcal{T}(d,h))}/\Lambda$ and that $\bar{\mathbf{v}} := \phi(\mathbf{v})$. Also, recall that V_j^q denotes the set of descendants of the vertex j, which are at depth q.

Lemma 6.4.1 *Let $j \in S_n$ $(1 \le n \le h)$.*
Then $\theta(d,h+2-n)\bar{\mathbf{x}}_j - \theta(d,h+1-n)\bar{\mathbf{x}}_{p(j)} = 0$.

Proof: Consider the sum

$$\sum_{q=n}^{h} [(d-1)^{h+1-q} - 1] \sum_{k \in V_j^q} \boldsymbol{\delta}_k. \tag{6.15}$$

Let $k \in V_j^q$ with $q > n$. The coefficient of $\mathbf{x}_k$ in sum (6.15) is

$$\begin{aligned} -(d-1)[(d-1)^{h-q} - 1] + d[(d-1)^{h+1-q} - 1] - [(d-1)^{h+2-q} - 1] &= \\ -(d-1)^{h+1-q} + d - 1 + d(d-1)^{h+1-q} - d - (d-1)^{h+2-q} + 1 &= \\ (d-1)(d-1)^{h+1-q} - (d-1)^{h+2-q} &= 0. \end{aligned}$$

The coefficient of $\mathbf{x}_j$ is

$$-(d-1)[(d-1)^{h-n} - 1] + d[(d-1)^{h+1-n} - 1] = (d-1)^{h+2-n} - 1.$$

Finally the coefficient of $\mathbf{x}_{p(j)}$ is

$$-[(d-1)^{h+1-n} - 1].$$

Therefore we have

$$\sum_{q=n}^{h}[(d-1)^{h+1-q}-1]\sum_{k\in V_j^q}\boldsymbol{\delta}_k = [(d-1)^{h+2-n}-1]\mathbf{x}_j - [(d-1)^{h+1-n}-1]\mathbf{x}_{p(j)},$$

and thus (dividing by $d-2$) we obtain

$$\sum_{q=n}^{h}\theta(d,h+1-q)\sum_{k\in V_j^q}\boldsymbol{\delta}_k = \theta(d,h+2-n)\mathbf{x}_j - \theta(d,h+1-n)\mathbf{x}_{p(j)}, \tag{6.16}$$

which implies

$$\theta(d,h+2-n)\bar{\mathbf{x}}_j - \theta(d,h+1-n)\bar{\mathbf{x}}_{p(j)} = 0.$$

□

Proposition 6.4.2 *Let* $\mathbf{x}_0$ *be the generator corresponding to the root* 0. *Then* $\mathrm{ord}(\bar{\mathbf{x}}_0) = d(d-1)^h$.

Proof: Let $j \in C_0 = \{1,\ldots,d\}$. By equation (6.16) for $n := 1$ we have:

$$\sum_{q=1}^{h}\theta(d,h+1-q)\sum_{k\in V_j^q}\boldsymbol{\delta}_k = \theta(d,h+1)\mathbf{x}_j - \theta(d,h)\mathbf{x}_0.$$

Adding the above d equations we obtain

$$\sum_{q=1}^{h}\theta(d,h+1-q)\sum_{k\in S_q}\boldsymbol{\delta}_k = \theta(d,h+1)\sum_{j\in C_0}\mathbf{x}_j - d\theta(d,h)\mathbf{x}_0. \tag{6.17}$$

From the definition of $\boldsymbol{\delta}_0$ we obtain

$$\theta(d,h+1)\boldsymbol{\delta}_0 = \theta(d,h+1)(d\mathbf{x}_0 - \sum_{j\in C_0}\mathbf{x}_j). \tag{6.18}$$

Adding equation (6.18) to equation (6.17) and using the definition of the $\theta(d,n)$ (Formula (1.1)) we obtain

$$\sum_{q=0}^{h} \theta(d,h+1-q) \sum_{k \in S_q} \boldsymbol{\delta}_k = [d/(d-2)][(d-1)^{h+1} - 1 - (d-1)^h + 1]\mathbf{x}_0$$

and thus

$$\sum_{q=0}^{h} \theta(d,h+1-q) \sum_{k \in S_q} \boldsymbol{\delta}_k = d(d-1)^h \mathbf{x}_0.$$

By Lemma 6.1.2 (noting that $\theta(d,1)=1$) we obtain $\mathrm{ord}(\bar{\mathbf{x}}_0) = d(d-1)^h$. □

Lemma 6.4.3 *Let ℓ be a leaf. Then*

$$\mathrm{ord}(\bar{\mathbf{x}}_\ell) \mid (d-1)^h \cdot \mathrm{lcm}\{d\theta(d,h+1), \theta(d,h), \theta(d,h-1), \ldots, \theta(d,2)\}.$$

Proof: Let $\ell = k_h, k_{h-1}, \ldots, k_0 = 0$ be the path from ℓ to the root 0. By Lemma 6.4.1 we have that

$$\theta(d,h-n+1)\bar{\mathbf{x}}_{k_{n+1}} = \theta(d,h-n)\bar{\mathbf{x}}_{k_n} \ (0 \le n \le h-1),$$

and by Lemma 6.4.2 we have that

$$\mathrm{ord}(\bar{\mathbf{x}}_0) = d(d-1)^h.$$

Apply Lemma 6.1.4 with $t := h$, $\mathbf{z}_n := \bar{\mathbf{x}}_{k_n}$ $(0 \le n \le h)$, $r_n := \theta(d,h+1-n)$ $(0 \le n \le h)$ and $s := d(d-1)^h$ to obtain

$$\mathrm{ord}(\bar{\mathbf{x}}_h) \mid \mathrm{lcm}\{(d-1)^h d\theta(d,h+1), \theta(d,h), \ldots, \theta(d,2)\}. \tag{6.19}$$

The numbers d and $\theta(d,n)$ are all relatively prime to $(d-1)^h$ and the conclusion follows. □

Lemma 6.4.4 *Let ℓ be a leaf. Then the exponent of $G(d,h)$ is the order of* $\bar{\mathbf{x}}_\ell$.

Proof: Let $f = f(d,h) := \mathrm{ord}(\bar{\mathbf{x}}_\ell)$ for all leaves ℓ. We show that for every vertex i, $f\bar{\mathbf{x}}_i = 0$, using induction on the distance of i from the boundary. If $i \in S_{h-1}$, then $d\bar{\mathbf{x}}_{\ell_0} = \bar{\mathbf{x}}_i$ for some leaf $\ell_0 \in V_i^h$ and so $f\bar{\mathbf{x}}_i = 0$. Now, let $i \in S_{h-n}$ and let $m \in C_i$. The relation $f\bar{\mathbf{x}}_i = 0$ follows from the defining relation $\bar{\mathbf{x}}_i = d\bar{\mathbf{x}}_m - \sum_{j\in C_m} \bar{\mathbf{x}}_j$ and the inductive hypothesis $f\bar{\mathbf{x}}_m = 0$ and $(\forall j \in C_m)(f\bar{\mathbf{x}}_j = 0)$. □

Lemmas 6.4.3 and 6.4.4 yield an upper bound for $\exp(d,h)$:

Proposition 6.4.5 *The exponent* $\exp(d,h)$ *of* $G(d,h)$ *satisfies*

$$\exp(d,h) \mid (d-1)^h \cdot \mathrm{lcm}\{d\theta(d,h+1), \theta(d,h), \theta(d,h-1), \ldots, \theta(d,2)\}.$$

Proposition 6.4.6 *Let $j_1, j_2 \in S_n$ be siblings $(1 \le n \le h)$. Then the order of $\bar{\mathbf{x}}_{j_1} - \bar{\mathbf{x}}_{j_2}$ is $\theta(d,h+2-n)$.*

Proof: Using equation (6.16) for j_1 and j_2, we obtain:

$$\theta(d,h+2-n)(\mathbf{x}_{j_1} - \mathbf{x}_{j_2}) = \sum_{q=n}^{h} \theta(d,h+1-q) \left[\sum_{k\in V_{j_1}^q} \boldsymbol{\delta}_k - \sum_{k\in V_{j_2}^q} \boldsymbol{\delta}_k \right].$$

By Lemma 6.1.2 (noting that $\theta(d,1)=1$) we obtain $\mathrm{ord}(\bar{\mathbf{x}}_{j_1} - \bar{\mathbf{x}}_{j_2}) = \theta(d,h+2-n)$. □

Proof of Theorem 1.5.7: By Propositions 6.4.2 and 6.4.6 we have

$$(d-1)^h \mathrm{lcm}\{\theta(d,n) : n = 1,\ldots,h\} \mid \exp(d,h)$$

Now we need to show that $d\theta(d,h+1) \mid \exp(d,h)$. Let $p \mid d\theta(d,h+1)$.

If $p \nmid d$ then $e_p(d\theta(d, h+1)) = e_p(\theta(d, h+1))$ and the result follows from Proposition 6.4.6. If $p \nmid \theta(d, h+1)$ then $e_p(d\theta(\text{d,h+1})) = e_p(d)$ and the result follows from Proposition 6.4.2. If $p \mid d$ and $p \mid \theta(d, h+1)$, in the Sylow p-subgroup of $G(d, h)$ consider the equation obtained by Lemma 6.4.1 for $n := 1$:

$$e_p(\theta(d, h+1))\bar{\mathbf{x}}_1 = e_p(\theta(d, h))\bar{\mathbf{x}}_0. \tag{6.20}$$

We have $p \nmid \theta(d, h)$. By equation (6.20) and by Proposition 6.4.2 we obtain $e_p(\bar{\mathbf{x}}_1) = e_p(d\theta(d, h+1))$. □

The next Proposition is interesting in its own right:

Proposition 6.4.7 *Let* $\mathbf{y}_n = \mathbf{y}(d, n) := \sum_{i \in S_n} \mathbf{x}_i$ $(1 \le n \le h)$. *Then the order of* $\bar{\mathbf{y}}_n$ *is* $(d-1)^{h+1-n}$.

Proof: Adding all the vectors $\boldsymbol{\delta}_k$ yields

$$(d-1)\mathbf{y}_h = \sum_{q=0}^{h} \sum_{k \in S_q} \boldsymbol{\delta}_k.$$

By Lemma 6.1.2 we have $\operatorname{ord}(\bar{\mathbf{y}}_h) = d - 1$.

We proceed by induction on the distance from the boundary. Assume

$$\operatorname{ord}(\bar{\mathbf{y}}_{n+1}) = (d-1)^{h-n} \tag{6.21}$$

and observe that

$$(d-1)\mathbf{y}_n - \mathbf{y}_{n+1} = \sum_{q=0}^{n} \sum_{k \in S_q} \boldsymbol{\delta}_k,$$

so

$$(d-1)\bar{\mathbf{y}}_n = \bar{\mathbf{y}}_{n+1}. \tag{6.22}$$

By equations (6.21) and (6.22) we obtain $\operatorname{ord}(\bar{\mathbf{y}}_n) = (d-1)^{h-n+1}$. □

6.5 The order and Hall subgroup

In this section we prove the order formula (1.3):

$$|G(d,h)| = d(d-1)^h[\theta(d,h+1)]^{d-1}\prod_{n=1}^{h-1}[\theta(d,h+1-n)]^{(d-2)d(d-1)^{n-1}}.$$

Moreover, we prove that the $(d-1)$ Hall-subgroup of $G(d,h)$ is cyclic of order $(d-1)^h$ (Theorem 1.5.4).

Definition 6.5.1 Let $G_n(d,h)$ be the subgroup of $G(d,h)$ generated by $\{\bar{\mathbf{x}}_i : i \in B_n\}$ $(0 \le n \le h)$.

Notation 6.5.2 Let $B \subseteq V(\mathcal{T}(d,h))$. Denote by pr_B the projection map from $\mathbb{Z}^{V(\mathcal{T}(d,h))}$ to $\oplus_{i\in B}\mathbb{Z}\mathbf{x}_i$. So, for $\mathbf{v} = (v_i) \in \mathbb{Z}^{V(\mathcal{T}(d,h))}$ we have $\mathrm{pr}_B(\mathbf{v}) := (v_i)_{i\in B}$.

Definition 6.5.3 Let $j \in S_n$ $(0 \le n \le h-1)$.

(i) Let $\mathcal{S}_j(d,h)$ denote the subgraph of $\mathcal{T}(d,h)$ induced on the set V_j of descendants of j. Note that $\mathcal{S}_0(d,h) = \mathcal{T}(d,h)$. Adjoin a "sink" to $V_j = V(\mathcal{S}_j(d,h))$ and if $n \ge 1$, then join j and every leaf of $\mathcal{S}_j(d,h)$ with 1 and $d-1$ edges respectively to the sink. Let $\hat{\mathcal{S}}_j(d,h)$ denote this augmented graph. If $n = 0$, i. e., if j is the root 0, let $\hat{\mathcal{S}}_j(d,h) := \hat{\mathcal{T}}(d,h)$.

(ii) Let $M_j(d,h)$ be the sandpile group of $\hat{\mathcal{S}}_j(d,h)$.

(iii) Let $K_j(d,h) := M_j(d,h)/\langle\bar{\mathbf{x}}_j\rangle$, where $\langle\bar{\mathbf{x}}_j\rangle$ is the cyclic subgroup of $M_j(d,h)$ generated by $\bar{\mathbf{x}}_j$, where, for the purposes of this definition, $\bar{\mathbf{x}}_j$ denotes the image of $\mathrm{pr}_{V_j}(\mathbf{x}_j)$ in the sandpile group $M_j(d,h)$.

Lemma 6.5.4 *Let j be a vertex. The vectors $\{\mathrm{pr}_{V_j}(\boldsymbol{\delta}_k) : k \in V_j\}$ are linearly independent over $\mathbb{Q}$.*

Proof: The vectors $\{\mathrm{pr}_{V_j}(\boldsymbol{\delta}_k) : k \in V_j\}$ are the row vectors of the reduced Laplacian matrix of $\hat{\mathcal{S}}_j(d,h)$. Therefore

$$M_j(d,h) \simeq \bigoplus_{k \in V_j} \mathbb{Z}\mathrm{pr}_{V_j}(\mathbf{x}_k) \Big/ \sum_{k \in V_j} \mathbb{Z}\mathrm{pr}_{V_j}(\boldsymbol{\delta}_k)\ , \tag{6.23}$$

and since $M_j(d,h)$ is finite the result follows. □

Definition 6.5.5 Let j be a vertex and let $B \subseteq V(\mathcal{T}(d,h))$. Let $\Gamma(j,B)$ be the vector subspace of $\mathbb{Q}^B$ generated by the $\{\mathrm{pr}_B(\boldsymbol{\delta}_k) : k \in V_j\}$.

Lemma 6.5.6 *Let $j \in S_n$ $(0 \leq n \leq h)$ and let $A \subseteq V_j$, $|A| = |V_j| - 1$. Then $\{\mathrm{pr}_{V_j \setminus \{j\}}(\boldsymbol{\delta}_k) : k \in A\}$ is linearly independent over $\mathbb{Q}$.*

Proof: We have

$$\Gamma(j, V_j \setminus \{j\}) \simeq \Gamma(j, V_j)/\mathbb{Q}\mathrm{pr}_{V_j}(\mathbf{x}_j)\ (0 \leq n \leq h).$$

By Lemma (6.5.4) we have

$$\dim(\Gamma(j, V_j)) = |V(j)|\ (0 \leq n \leq h), \tag{6.24}$$

so

$$\dim(\Gamma(j, V_j \setminus \{j\})) = |V_j| - 1. \tag{6.25}$$

Recall that C_j is the set of children of j. Now let us consider equation (6.16). Let us replace j by m and n by $n+1$; and let us apply $\mathrm{pr}_{V_j \setminus \{j\}}$ to (6.16) with this modified notation. Noting that $p(m) = j$ and therefore $\mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_{p(m)}) = \mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_j) = 0$ we then obtain

$$\theta(d,h+1-n)\mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_m) = \sum_{q=n+1}^{h} \theta(d,h+1-q)\left[\sum_{k\in V_m^q} \mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_k)\right] \ (m \in C_j). \tag{6.26}$$

Also, by the definition of δ_j we have

$$\mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_j) + \sum_{m\in C_j} \mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_m) = 0. \tag{6.27}$$

Multiplying equation (6.27) by $\theta(d,h+1-n)$ and using equation (6.26) for all $m \in C_j$ we obtain

$$\theta(d,h+1-n)\mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_j) + \sum_{m\in C_j}\sum_{q=n+1}^{h} \theta(d,h+1-q)\left[\sum_{k\in V_m^q} \mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_k)\right] = 0. \tag{6.28}$$

Noting that all the coefficients $\theta(d,h+1-q)$ in (6.28) are non-zero, equations (6.25) and (6.28) show that omitting any of the vectors from the set $\{\mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_k) : k \in V_j\}$, the rest is a basis of $\Gamma(j, V_j\backslash\{j\})$. □

Definition 6.5.7 Let $j \in V(\mathcal{T}(d,h))$. Let us pick a leaf $\ell_j \in V_j^h$. Set $U_j := V_j\backslash\{\ell_j\}$.

The next Lemma describes $|V_j^h|$ distinct sets of relations, each of which defines $K_j(d,h)$.

Lemma 6.5.8 *Let* $j \in S_n$ $(0 \le n \le h-1)$*. Then*

$$K_j(d,h) \simeq \bigoplus_{k\in V_j\backslash\{j\}} \mathbb{Z}\mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_k) \Big/ \sum_{k\in U_j} \mathbb{Z}\mathrm{pr}_{V_j\backslash\{j\}}(\boldsymbol{\delta}_k)\ .$$

Proof: By equation (6.23) and by the definition of $K_j(d,h)$ we have

$$K_j(d,h) \simeq \bigoplus_{k\in V_j} \mathbb{Z}\mathrm{pr}_{V_j}(\mathbf{x}_k) \Big/ \left[\sum_{k\in V_j} \mathbb{Z}\mathrm{pr}_{V_j}(\boldsymbol{\delta}_k) + \mathbb{Z}\mathrm{pr}_{V_j}(\mathbf{x}_j)\right] \simeq$$

$$\simeq \bigoplus_{k\in V_j\setminus\{j\}} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_k) \Big/ \sum_{k\in V_j} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k)\,. \quad (6.29)$$

By equation (6.28) (noting that $\theta(d,1)=1$) we have

$$\sum_{k\in V_j} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k) = \sum_{k\in U_j} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k). \quad (6.30)$$

Now combine equation (6.29) with equation (6.30). □

Lemma 6.5.9 $G(d,h)/G_n(d,h) \simeq \bigoplus_{j\in S_n} K_j(d,h)$ $(0 \le n \le h-1)$.

Proof: Set $V := V(\mathcal{T}(d,h))$. We have

$$G(d,h)/G_n(d,h) \simeq \bigoplus_{j\in V} \mathbb{Z}\mathbf{x}_j \Big/ \left[\sum_{j\in V} \mathbb{Z}\boldsymbol{\delta}_j + \sum_{j\in B_n} \mathbb{Z}\mathbf{x}_j\right] \simeq$$

$$\simeq \bigoplus_{j\in V\setminus B_n} \mathbb{Z}\mathrm{pr}_{V\setminus B_n}(\mathbf{x}_j) \Big/ \left[\sum_{j\in V\setminus B_{n-1}} \mathbb{Z}\mathrm{pr}_{V\setminus B_n}(\boldsymbol{\delta}_j)\right]. \quad (6.31)$$

Let $j_1, j_2 \in S_n$ with $j_1 \neq j_2$ and let $j_1 \preceq k_1$ and $j_2 \preceq k_2$. Then the vectors $\mathrm{pr}_{V\setminus B_n}(\boldsymbol{\delta}_{k_1})$ and $\mathrm{pr}_{V\setminus B_n}(\boldsymbol{\delta}_{k_2})$ are disjoint. Therefore by equation (6.31) we obtain

$$G(d,h)/G_n(d,h) \;\simeq\; \bigoplus_{j\in S_n} \left[\bigoplus_{k\in V_j\setminus\{j\}} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_k) \Big/ \sum_{k\in V_j\setminus\{j\}} \mathbb{Z}\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k)\right]. \quad (6.32)$$

By equations (6.29) and (6.32) we obtain the result. □

Lemma 6.5.10 *Let $j \in S_n$ and let $i := p(j)$ be the parent of j. Then the order of $\mathrm{pr}_{V_i \setminus \{i\}}(\mathbf{x}_j)$ in $K_i(d,h)$ is $\theta(d, h+2-n)$ $(1 \le n \le h)$.*

Proof: Let us apply $\mathrm{pr}_{V_i \setminus \{i\}}$ to equation (6.16). Noting that $\mathrm{pr}_{V_i \setminus \{i\}}(\mathbf{x}_{p(j)}) = \mathrm{pr}_{V_i \setminus \{i\}}(\mathbf{x}_i) = 0$ we obtain

$$\theta(d, h+2-n)\mathrm{pr}_{V_i \setminus \{i\}}(\mathbf{x}_j) = \sum_{q=n}^{h} \theta(d, h+1-q) \left[\sum_{k \in V_j^q} \mathrm{pr}_{V_i \setminus \{i\}}(\boldsymbol{\delta}_k) \right]. \quad (6.33)$$

In equation (6.33) we have obtained an expression of $\theta(d, h+2-n)\mathrm{pr}_{V_i \setminus \{i\}}(\mathbf{x}_j)$ as an integer linear combination of the vectors $\{\mathrm{pr}_{V_i \setminus \{i\}}(\boldsymbol{\delta}_k) : k \in V_j\}$. Let j' be a sibling of j and let $\ell_{j'} \in V_{j'}^h \subseteq V_i^h$ be a leaf below j'. Then $V_j \subseteq V_i \setminus \{\ell_{j'}\}$. By Lemma 6.5.8 we have that $\{\mathrm{pr}_{V_i \setminus \{i\}}(\boldsymbol{\delta}_k) : k \in V_j\}$ is part of a lattice basis that defines $K_i(d,h)$. Noting that $\theta(d,1) = 1$, the conclusion follows from Lemma 6.1.2. □

Proposition 6.5.11 *Let $j \in S_n$. Then the order of $\mathbf{x}_j$ in $G(d,h)/G_{n-1}(d,h)$ is $\theta(d, h+2-n)$ $(1 \le n \le h)$.*

Proof: The conclusion follows from Lemmas 6.5.9 and 6.5.10. □

Lemma 6.5.12 *Let $j \in S_n$ $(0 \le n \le h-1)$. Let $C'_j \subseteq C_j$ with $|C'_j| = |C_j| - 1$. Assume*

$$\sum_{m \in C'_j} r_m \mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_m) = 0 \text{ in } K_j(d,h). \quad (6.34)$$

Then $r_m \mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_m) = 0$ in $K_j(d,h)$ $(m \in C'_j)$.

Proof: Assume for a contradiction that for some $m_0 \in C'_j$ we have

$$r_{m_0} \mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_{m_0}) \neq 0 \text{ in } K_j(d,h). \quad (6.35)$$

Lemma 6.5.10 applied with j replaced by m_0, i replaced by j and n replaced by $n+1$ implies that

$$\theta(d,h+1-n)\mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_{m_0}) = 0 \text{ in } K_j(d,h). \tag{6.36}$$

By equations (6.35) and (6.36) we have

$$\theta(d,h+1-n) \nmid r_{m_0}. \tag{6.37}$$

Let $C_j'' := C_j'\backslash\{m_0\}$. By the Euclidean algorithm for r_{m_0} and $\theta(d,h+1-n)$ and by equations (6.34) and (6.36) we obtain

$$\sum_{m\in C_j'} s_m \mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_m) = 0 \text{ in } K_j(d,h), \tag{6.38}$$

where

$$s_{m_0} = \gcd(r_{m_0}, \theta(d,h+1-n)) \text{ and } s_m \in \mathbb{Z}\ (m \in C_j''). \tag{6.39}$$

Set

$$t_1 := \gcd\{s_m : m \in C_j'\}, \tag{6.40}$$

and

$$t_2 := \gcd\{s_m\theta(d,h+1-q) : m \in C_j', q = n+1,\ldots,h\}. \tag{6.41}$$

Claim 6.5.13 $t_1 = t_2$.

Proof of Claim 6.5.13: Clearly $t_1 \mid t_2$. For the other direction consider the definition (6.41). Setting $q := h$ and recalling that $\theta(d,1) = 1$ we obtain

$$t_2 \mid s_m\ (m \in C_j'),$$

and therefore $t_2 \mid t_1$. □

Observe that $s_{m_0} \mid \theta(d,h+1-n)$ and therefore $t_1 \mid \theta(d,h+1-n)$.

Claim 6.5.14 *The order of* $\sum_{m\in C_j'} s_m \mathrm{pr}_{V_j\backslash\{j\}}(\mathbf{x}_m)$ *in* $K_j(d,h)$ *is equal to* $\theta(d,h+1-n)/t_1$.

Proof of Claim 6.5.14: We have

$$\theta(d,h+1-n)/t_1 \sum_{m\in C'_j} s_m \mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_m) =$$
$$= \sum_{m\in C'_j} (s_m/t_1)\theta(d,h+1-n)\mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_m). \quad (6.42)$$

Substituting $\theta(d,h+1-n)\mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_m)$ from equation (6.26) into (6.42) we obtain

$$\theta(d,h+1-n)/t_1 \sum_{m\in C'_j} s_m \mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_m) =$$
$$= \sum_{m\in C'_j} s_m/t_1 \sum_{q=n+1}^{h} \theta(d,h+1-q) \sum_{k\in V_m^q} \mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k) =$$
$$= \sum_{m\in C'_j} \sum_{q=n+1}^{h} s_m\theta(d,h+1-q)/t_1 \sum_{k\in V_m^q} \mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k). \quad (6.43)$$

Let $\{m_1\} = C_j\setminus C'_j$ and let $\ell \in V_{m_1}^h$ be a leaf below m_1. In equation (6.43) we have obtained an expression of the vector $\theta(d,h+1-n)/t_1 \sum_{m\in C'_j} s_m \mathrm{pr}_{V_j\setminus\{j\}}(\mathbf{x}_m)$ as a linear combination of the vectors

$$\{\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k) : k \in \bigcup_{m\in C'_j} V_m\} \subseteq \{\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k) : k \in V_j\setminus\{\ell\}\}.$$

By Lemma 6.5.8 the set $\{\mathrm{pr}_{V_j\setminus\{j\}}(\boldsymbol{\delta}_k) : k \in \bigcup_{m\in C'_j} V_m\}$ is part of a lattice basis that defines $K_j(d,h)$. Also, by Claim 6.5.13 we have

$$\gcd\{s_m\theta(d,h+1-q)/t_1 : m \in C'_j, q = n+1,\ldots,h\} =$$
$$= (1/t_1)\gcd\{s_m\theta(d,h+1-q) : m \in C'_j, q = n+1,\ldots,h\} = t_2/t_1 = 1,$$

so, the coefficients in equation (6.43) are relatively prime. The conclusion follows by Lemma 6.1.2. □

By equation (6.38) and by Claim 6.5.14 we conclude

$$\theta(d, h+1-n) = t_1 = \gcd\{s_m : m \in C'_j\}.$$

Therefore $\theta(d, h+1-n) \mid s_{m_0}$ and hence by equation (6.39), $\theta(d, h+1-n) \mid r_{m_0}$, which contradicts our original assumption (equation (6.37)). □

Definition 6.5.15 Let $j \in S_n$ $(0 \leq n \leq h-1)$. Let $E_j(d,h)$ be the subgroup of $K_j(d,h)$ generated by the images of the $\{\mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_m) : m \in C_j\}$ in $K_j(d,h)$ represented as in Lemma 6.5.8.

Lemma 6.5.16 *Let $j \in S_n$. Then*

$$E_j \simeq \underbrace{\mathbb{Z}_{\theta(d,h+1-n)} \oplus \cdots \oplus \mathbb{Z}_{\theta(d,h+1-n)}}_{|C_j|-1 \text{ times}} \ (0 \leq n \leq h-1).$$

Proof: By Lemma 6.5.8 we have that $\mathrm{pr}_{V_j \setminus \{j\}}(\boldsymbol{\delta}_j)$ is a member of a lattice basis that defines $K_j(d,h)$. Therefore

$$\mathrm{pr}_{V_j \setminus \{j\}}(\boldsymbol{\delta}_j) = 0 \text{ in } K_j(d,h). \tag{6.44}$$

By equations (6.27) and (6.44) we obtain

$$\sum_{m \in C_j} \mathrm{pr}_{V_j \setminus \{j\}}(\mathbf{x}_m) = 0 \text{ in } K_j(d,h). \tag{6.45}$$

Equation (6.45) combined with Lemmas 6.5.10 (applied with j replaced by m, i replaced by j and n replaced by $n+1$) and 6.5.12 yield the result. □

Lemma 6.5.17

$$G_{n+1}(d,h)/G_n(d,h) \simeq \bigoplus_{j \in S_n} E_j \ (0 \leq n \leq h-1).$$

Proof: The result follows from Lemma 6.5.9. □

Lemma 6.5.18

$$|G_{n+1}(d,h)/G_n(d,h)| =$$

$$= \begin{cases} \theta(d,h+1)^{d-1} & \text{if } n=0, \\ \theta(d,h+1-n)^{(d-2)d(d-1)^{n-1}} & \text{if } 1 \le n \le h-1. \end{cases}$$

Proof: The conclusion follows from Lemmas 6.5.16 and 6.5.17. □

Proof of Theorem 1.5.9: Consider the chain of subgroups $G_0(d,h) \le G_1(d,h) \le \ldots \le G_h(d,h) = G(d,h)$. Clearly,

$$|G(d,h)| = |G_0(d,h)| \prod_{n=0}^{h-1} |G_{n+1}(d,h)/G_n(d,h)|.$$

Proposition 6.4.2 and Lemma 6.5.18 give the result. □

Lemma 6.5.19 *$G(d,h)$ has a cyclic subgroup of order $(d-1)^h$.*

Proof: By Proposition 6.4.2 we have that $\mathrm{ord}(d\bar{\mathbf{x}}_0) = (d-1)^h$. □

Proof of Theorem 1.5.4: We have that d and the numbers $\theta(d,j)$ are relatively prime to $d-1$. This fact combined with Lemma 6.5.19 and with the order formula (1.3) (Theorem 1.5.9) proves Theorem 1.5.4. □

6.6 Asymptotic evaluation

Proof of Corollary 1.5.10: Formula (1.3) yields

$$\begin{aligned}\log_{d-1} g(d,h) &= \log_{d-1} d + h + (d-1)\log_{d-1}\theta(d,h+1) + \\ &+ \sum_{n=1}^{h-1}(d-2)d(d-1)^{n-1}\log_{d-1}\left(\frac{(d-1)^{h+1-n}-1}{d-2}\right) = \\ &= \log_{d-1} d + h + (d-1)\log_{d-1}\theta(d,h+1) + \\ &+ \sum_{n=0}^{h-2}(d-2)d(d-1)^{h-n-2}\log_{d-1}\left(\frac{(d-1)^{n+2}-1}{d-2}\right).\end{aligned}$$

Clearly,

$$\lim_{h\to\infty}\frac{\log_{d-1} g(d,h)}{(d-1)^h} = c_d.$$

□

Notation 6.6.1

Let r, s be positive integers. Set $\eta(r,s) := \operatorname{lcm}\{r^n - 1 : n = 1,\dots,s\}$.

We will need the following estimate (see [30]):

Theorem 6.6.2 *For every fixed $r \geq 2$, the following asymptotic equality holds as $s \to \infty$:*

$$\log_r \eta(r,s) \sim \frac{3s^2}{\pi^2}.$$

Proof of Corollary 1.5.8: By formula (1.2) we obtain

$$\eta(d-1,h+1) \leq \exp(d,h) \leq d(d-1)^h\eta(d-1,h+1). \tag{6.46}$$

The result follows by combining formula (6.46) with Theorem 6.6.2 used with parameters $r := d-1$ and $s := h+1$. □

CHAPTER 7
FURTHER DIRECTIONS

We have calculated the exponent and the rank of the sandpile group $G(d,h)$ of the complete d-regular tree of depth h. Finding all invariants explicitly, i. e., fully determining the group structure, is equivalent to knowing the structure of all Sylow subgroups. We now describe an explicit conjecture regarding the rank of each Sylow subgroup.

Notation 7.0.3 Let p be a prime. We denote by $S_p(d,h)$ the Sylow p-subgroup of $G(d,h)$.

If $p \mid d$ then by Lemma 6.3.8 we have $\operatorname{rank}(S_p(d,h)) = (d-1)^h$. If $p \mid d-1$ then by Theorem 1.5.4 we have $\operatorname{rank}(S_p(d,h)) = 1$.

Notation 7.0.4

(i) Let r, s be relatively prime integers with $s > 0$. We denote by $\operatorname{ord}_s(r)$ the multiplicative order of r in the ring $\mathbb{Z}/s\mathbb{Z}$.

(ii) Let p be a prime with $p \nmid d-1$. We denote by $t_p(d)$ the least positive integer n such that $p \mid \theta(d,n)$ (see Definition 1.1). Note that

$$t_p(d) = \begin{cases} p & \text{if } d \equiv 2 \mod p, \\ \operatorname{ord}_p(d-1) & \text{otherwise.} \end{cases}$$

Conjecture 7.0.5 *Let $p \nmid d(d-1)$. The rank of the Sylow p-subgroup $S_p(d,h)$ of $G(d,h)$ is given by the following formula:*

$$\operatorname{rank}(S_p(d,h)) = \begin{cases} d(d-1)^{h-rt_p(d)} \sum_{q=0}^{r-1} (d-1)^{qt_p(d)} \\ \qquad\qquad \text{if } rt_p(d) \leq h \leq (r+1)t_p(d) - 2, \\ \\ d(d-1)^{t_p(d)-1} \sum_{q=0}^{r-1} (d-1)^{qt_p(d)} + d - 1 \\ \qquad\qquad \text{if } h = (r+1)t_p(d) - 1. \end{cases} \tag{7.1}$$

The structure of the Sylow subgroups remains a mystery, even for primes dividing d. The following data for the Sylow 3-subgroups for the trivalent trees ($d = 3$) of depths $h = 1, \ldots, 8$ does not seem to reveal a pattern as to how the rank 2^h is partitioned.

h	$\mathrm{Syl}_3(G(3,h))$	$\mathrm{rank}(\mathrm{Syl}_3(G(3,h)))$
1	$\mathbb{Z}_3 \oplus \mathbb{Z}_9$	2
2	$\mathbb{Z}_3^4$	4
3	$\mathbb{Z}_3^7 \oplus \mathbb{Z}_9$	8
4	$\mathbb{Z}_3^{16}$	16
5	$\mathbb{Z}_3^{30} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27}$	32
6	$\mathbb{Z}_3^{61} \oplus \mathbb{Z}_9^3$	64
7	$\mathbb{Z}_3^{121} \oplus \mathbb{Z}_9^7$	128
8	$\mathbb{Z}_3^{244} \oplus \mathbb{Z}_9^{12}$	256

Table 7.1: The Sylow 3-subgroup of $G(3,h)$ for $h = 1, \ldots, 8$

APPENDIX A
NOTATION FOR CHAPTER 6

$\mathcal{T}(d) = (V(\mathcal{T}(d)), E(\mathcal{T}(d)))$	infinite d-regular tree
$\mathcal{P}$	subgraph induced on $P \subseteq V(\mathcal{T}(d))$
$\hat{\mathcal{P}}$	$\mathcal{P}$ with sink adjoined
$\mathcal{T}(d, h)$	complete d-regular tree of depth h
$\hat{\mathcal{T}}(d, h)$	$\mathcal{T}(d, h)$ with sink adjoined
$\mathcal{S}_j(d, h)$	the subgraph of $\mathcal{T}(d, h)$ induced on V_j (see Definition 6.5.3)
$\hat{\mathcal{S}}_j(d, h)$	$\mathcal{S}_j(d, h)$ with "sink" adjoined (see Definition 6.5.3)

Table A.1: Notation: Graphs

i, j, k, m	vertices
0	the root
$1, \ldots, d$	the vertices at depth 1
m_i	some child of the vertex i
ℓ	a leaf
ℓ_i	some leaf that is descendant of the vertex i

Table A.2: Notation: Vertices

P	finite subset of $V(\mathcal{T}(d))$
P_n	sequence of subsets of $V(\mathcal{T}(d))$
N_i	set of neighbors of the vertex i
V_i	set of descendants of i
U_i	$V_i \backslash \{\ell_i\}$
C_i	set of children of i
J_i	$C_i \backslash \{m_i\}$
B_n	ball of radius n about the root
S_n	sphere of radius n about the root
V_i^n	$V_i \cap S^n$,i. e., descendants of i at depth n
V_i^h	set of leaves of $\mathcal{T}(d,h)$ that are descendants of i
$V := V(\mathcal{T}(d,h))$	the set of vertices of $\mathcal{T}(d,h)$
$F(d,h)$	subset of $V(\mathcal{T}(d,h))$ (see Definition 6.3.2)
B	subset of $V(\mathcal{T}(d,h))$

Table A.3: Notation: Sets of vertices

d	degree of the regular tree $\mathcal{T}(d)$
h	depth of finite tree $\mathcal{T}(d,h)$
$\exp(d,h)$	the exponent of $G(d,h)$
$f(d,h)$	the order of $\mathbf{x}_\ell$ for ℓ leaf
$\theta(d,n)$	$[(d-1)^n - 1]/(d-2)$
$\eta(r,s)$	$\text{lcm}\{r^n - 1 : n = 1, \ldots, s\}$
c_d	a constant (see Corollary 1.5.10)

Table A.4: Notation: Constants

p, q, r, s, t	integers
n	integer usually between 0 and h

Table A.5: Notation: Integers

L	Laplacian of $\mathcal{X}^*$ (see Definition 3.2.14)
Δ	matrix obtained from L after deleting row and column corresponding to the sink

Table A.6: Notation: Matrices

G_n	sandpile group of $\hat{\mathcal{P}}_n$
$G(d,h)$	sandpile group associated with B_h
Λ	lattice in $\mathbb{Z}^t$ spanned by t linearly independent elements of $\mathbb{Z}^t$
K	$\mathbb{Z}^t/\Lambda$
$G_F(d,h)$	subgroup of $G(d,h)$ generated by $\{\bar{\mathbf{x}}_i : i \in F(d,h)\}$
$G_n(d,h)$	subgroup of $G(d,h)$ generated by $\{\bar{\mathbf{x}}_i : i \in B_n\}$
$M_j(d,h)$	sandpile group of $\hat{\mathcal{S}}_j(d,h)$ (see Definition 6.5.3)
$K_j(d,h)$	$M_j(d,h)/\langle \bar{\mathbf{x}}_j \rangle$ (see Definition 6.5.3)
H	Hall subgroup of a group G
$S_p(d,h)$	the Sylow p-subgroup of $G(d,h)$

Table A.7: Notation: Groups

$\boldsymbol{\delta}_i$	the row vector of Δ corresponding to the vertex i
$\mathbf{x}_i$	standard basis vector of $\mathbb{Z}^{V(\mathcal{T}(d,h))}$ corresponding to i
$\mathbf{v} = (v_i)_{i \in V(\mathcal{T}(d,h))}$	an element of $\mathbb{Z}^{V(\mathcal{T}(d,h))}$
$\bar{\mathbf{v}}$	$\phi(\mathbf{v})$ (see Table A.10)
$\mathbf{u}_1, \ldots, \mathbf{u}_t$	linearly independent elements of $\mathbb{Z}^t$
$\mathbf{u}$	some element in the lattice $\sum_{n=1}^t \mathbb{Z}\mathbf{u}_n$ generated by the $\mathbf{u}_n$
$\bar{\mathbf{u}}$	the image of u in the quotient group $\mathbb{Z}^t / \sum_{n=1}^t \mathbb{Z}\mathbf{u}_n$
$\mathbf{s}_h = (s_{h,i})_{i \in V(\mathcal{T}(d,h))}$	a solution of system (6.13)
$\bar{\mathbf{y}}_n = \bar{\mathbf{y}}(d,n) := \sum_{i \in S_n} \bar{\mathbf{x}}_i$	element of $G(d,h)$
$\mathbf{z}, \mathbf{z}_t$	elements of an abelian group G

Table A.8: Notation: Elements of groups-vectors

$\Gamma(j,B)$ the vector subspace of $\mathbb{Q}^B$ generated by the $\{\mathrm{pr}_B(\boldsymbol{\delta}_k) : k \in V_j\}$

Table A.9: Notation: Vector spaces

w_i	height of sandpile at vertex i
$\mathbf{w} = (w_i)_{i \in V(\mathcal{T}(d,h))}$	state
α_i	toppling operator corresponding to the vertex i
$p(i)$	the parent of the vertex i
$\omega : \mathbb{Z}^{V(\mathcal{T}(d,h))} \to \mathbb{Z}_d^{F(d,h)}$	see Equation (6.14)
ϕ	quotient map from $\mathbb{Z}^{V(\mathcal{T}(d,h))}$ to $G(d,h)$
$\deg_{\mathcal{P}}(i)$	degree of the vertex i in $\mathcal{P}$
$\mathrm{dist}(i,j)$	distance of the vertices i, j in $\mathcal{T}(d)$
$\mathrm{ord}(\mathbf{z})$	the order of $\mathbf{z} \in G$ in the group G
$e_p(s)$	$w \in \mathbb{Z}$, where $p^w \mid s$ and $p^{w+1} \nmid s$
$e_p(\mathbf{z})$	$e_p(\mathrm{ord}(\mathbf{z}))$
$\mathrm{pr}_B(\mathbf{v})$	$(v_i)_{i \in B}$
$\tilde{\mathbf{v}}$	$(v_i)_{i \in F(d,h)}$, i. e., $\tilde{\mathbf{v}} = \mathrm{pr}_{F(d,h)}(\mathbf{v})$
$\mathrm{ord}_s(r)$	the multiplicative order of r in $\mathbb{Z}_s$, ($\gcd(r,s) = 1$ and $s > 0$)
$t_p(d)$	the least positive integer n such that $p \mid \theta(d,n)$ ($p \nmid d - 1$)

Table A.10: Notation: Functions

REFERENCES

[1] S. R. Athreya, A. A. Járai: Infinite Volume Limit for the Stationary Distribution of Abelian Sandpile Models. *Communications in Mathematical Physics* **249** (2004) pp. 197–213

[2] L. Babai, E. Toumpakari: A structure theory of the sandpile monoid for digraphs. Manuscript (2005)

[3] H. Bai: On the critical group of the n-cube. *Linear Algebra and its Applications* **369** (2003) pp. 251–261

[4] P. Bak, C. Tang, K. Wiesenfeld: Self-organized Criticality. *Phys. Rev. A* **38** (1988) pp. 364–374

[5] N. Biggs: Algebraic Potential Theory on Graphs. *Bull. London Math. Soc.* **29** (1997) pp. 641–682

[6] N. Biggs: Chip-Firing and the Critical Group of a Graph. *Journal of Algebraic Combinatorics* **9** (1999) pp. 25–45

[7] A. Bjørner, L. Lovász: Chip-firing games on directed graphs. *Journal of Algebraic Combinatorics* **1** (1992) pp. 305–328

[8] A. Bjørner, L. Lovász, P. Shor: Chip-firing games on graphs. *Europ. J. Combinatorics* **12** (1991) pp. 283–291

[9] R. Cori, D. Rossin: On the Sandpile Group of Dual Graphs. *Europ. J. Combinatorics* **21** (2000) pp. 447–459

[10] M. Creutz: Abelian Sandpile. *Computers in Physics* **5** (1991) pp. 198–203

[11] A. Dartois, F. Fiorenzi, F. Francini: Sandpile group on the graph $\mathcal{D}_n$ of the dihedral group. *Europ. J. Combinatorics* **24** (2003) pp. 815–824

[12] D. Dhar: Self-organized Critical State of Sandpile Automaton Models. *Phys. Rev. Lett.* **64** (1990) pp. 1613–1616

[13] D. Dhar, S. Majumdar: Abelian Sandpile Model on the Bethe lattice. *J.Phys. A: Math. Gen.* **23** (1990) pp. 4333-4350

[14] D. Dhar, P. Ruelle, S. Sen, D. Verma: Algebraic Aspects of Abelian Sandpile Models. *J. Phys. A* **28** (1995) no.4 pp. 805–831

[15] K. Eriksson: No polynomial bound for the chip firing game on directed graphs. *Proc. Amer. Math. Soc.* **112** (1989) pp. 1203–1205.

[16] P.A. Grillet: *Semigroups (Section IV.8).* Pure and Applied Mathematics **193**, M. Dekker (1995)

[17] P.A. Grillet: Nilsemigroups on trees. *Semigroup Forum* **43** (1991) pp. 187–201.

[18] J. van den Heuvel: Algorithmic aspects of a chip-firing game. *Combinatorics, Probability and Computing* **10** (2001) pp. 505–529

[19] E. V. Ivashkevich, D. V. Ktitarev, V. B. Priezzhev: Waves of topplings in an abelian sandpile. *Physica A* **209** (1994) pp. 347–360

[20] B. Jacobson, A. Niedermaier, V. Reiner: Critical groups for complete multipartite graphs and cartesian products of complete graphs. *Journal of Graph Theory* **44** no.3 (2003) pp. 231–250

[21] H. Jensen: *Self-Organized Criticality.* Cambridge Lecture Notes in Physics **10** (1998)

[22] G. Kirchhoff: Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanisher Ströme geführt wird. *Ann. Phys. Chem.* **72** (1847) pp. 497-508

[23] D. J. Lorenzini: A finite group attached to the Laplacian of a graph. *Discrete Math* **91** (1991) pp. 277–282

[24] L. Lovász: *Combinatorial Problems and Exercises.* 2nd Edition (1993) Akadémiai Kiadó, Budapest, and Elsevier

[25] L. Lovász, P. Winkler: Mixing of random walks and other diffusions on a graph. P. Rowlinson (ed.), *Surveys in Combinatorics.* London Math. Soc. Lecture Notes Series **218** Cambridge University Press (1995) pp. 119–154.

[26] S. MacLane, G. Birkhoff: *Algebra.* 3rd Edition (1988) Chelsea Publishing Company, New York, N. Y.

[27] C. Maes, F. Redig, E. Saada: The Abelian Sandpile Model on an infinite tree. *eprint arXiv:math-ph/0101005* (2001)

[28] S. N. Majumdar, D. Dhar: Equivalence of the abelian sandpile model and the $q \to 0$ limit of the Potts model. *Physica A* **185** (1992) pp. 129–145

[29] C. Moore, M. Nilsson: The computational complexity of sandpiles. *J. Stat. Phys.* **96** (1999) pp. 205–224

[30] R. Narasimhan: Logarithmic asymptotics for lcm$\{d^m - 1 : m = 1, \dots, n\}$. Notes by L. Babai and E. Toumpakari (2005)
http://www.cs.uchicago.edu/∼ laci/cyclo.pdf

[31] M. H. A. Newmann: On theories with a combinatorial definition of "equivalence." *Ann. Math.* **43** (1942) pp. 223–243

[32] V. B. Priezzhev: Structure of two-dimensional sandpile height probabilities. *J. Stat. Phys.* **74** (1994) pp. 955–979

[33] E. Speer: Asymmetric Abelian Sandpile Models. *J. Stat. Phys.* **71** (1993) pp. 61–74

[34] G. Tardos: Polynomial bound for a chip-firing game on graphs. *SIAM J. Disc. Math.* **1** no.3 (1988) pp. 397–398

[35] E. Toumpakari: On the sandpile group of regular trees.
To appear in *Europ. J. Combinatorics*

[36] W. T. Tutte: The dissection of equilateral triangles into equilateral triangles. *Proc. Cambridge Phil. Soc.* **44** (1948) pp. 463–482