On incidence coloring and star arboricity of graphs

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Abstract

In this note we show that the concept of incidence coloring introduced in [BM] is a special case of directed star arboricity, introduced in [AA]. A conjecture in [BM] concerning asmyptotics of the incidence coloring number is solved in the negative following an example in [AA]. We generalize Theorem 2.1 of [AMR] concerning the star arboricity of graphs to the directed case by a slight modification of their proof, to give the same asymptotic bound as in the undirected case. As a result, we get tight asymptotic bounds for the maximum incidence coloring number of a graph in terms of its degree.

1 Connection between star arboricity and incidence coloring

A star forest is a graph whose connected components are stars. A directed star forest is a graph whose connected components are directed stars (edges are directed out of the center). The star arboricity of an undirected graph G (introduced in [AK]) denoted $\mathrm{st}(G)$ is the smallest number of star forests needed to cover G. For a directed graph D, the directed star arboricity denoted $\mathrm{dst}(D)$, is the smallest number of directed star forests needed to cover D.

In [BM], the concept of incidence coloring of graphs is introduced. $\iota(G)$ is the smallest number of colors needed to color the incidences (incident vertex-edge pairs) of an undirected graph G so that *neighborly* incidences do not receive the same color. Two incidences (v, e) and (w, f) are said to be *neighborly* if (i) $v = w$, (ii) $e = f$, or (iii) $\{v, w\}$ is one of the edges e, f. If we think of an incidence pair as a directed edge, directed toward the vertex, we are coloring the edges of the symmetrically directed graph $S(G)$ (we replace each edge by both directed edges). Each color class is a directed star forest. This shows that $\iota(G) = \mathrm{dst}(S(G))$.

2 Graphs with large incidence coloring number

In [BM] it is conjectured that $\iota(G) \leq \Delta(G) + 2$, where $\Delta = \Delta(G)$ is the maximum degree of a vertex of G. Following an example in Section 3 of [AA], we consider the Paley graphs. Let p be a prime with $p \equiv 1 \pmod 4$. The vertex set of G is $\{1, 2, \ldots, p\}$ and $\{i, j\}$ is an edge precisely when $i - j$ is a quadratic residue modulo p. Following the analysis in [AA], we see that

$$\iota(G) \geq \Delta + \Omega(\log \Delta).$$

This proves the conjecture false and raises the question as to how big $\iota(G)$ can be in terms of $\Delta(G)$.

3 Upper bounds

Until now everything we have done works for multigraphs as well; from now on we will limit ourselves to simple graphs (either directed or undirected). Prop 5.1 in [AA] gives the bound of:

$$\iota(G) \leq \Delta + O(\Delta^{\frac{2}{3}} (\log \Delta)^{\frac{1}{3}}).$$

In the same paper, using the same method, for the undirected case, the authors get the same asymptotic bound of $\mathrm{st}(G) \leq \Delta/2 + O(\Delta^{\frac{2}{3}} (\log \Delta)^{\frac{1}{3}})$ which is later improved in [AMR] using the Lovász Local Lemma to give $\mathrm{st}(G) \leq \Delta/2 + O(\log \Delta)$, a tight upper bound.

We show that their proof can be followed for the directed case, giving a tight upper bound:

$$\mathrm{dst}(D) \leq \Delta + O(\log \Delta),$$

where D is a directed graph with indegree and outdregree both less than Δ. For the edge-incidence case, this gives:

$$\iota(G) \leq \Delta + O(\log \Delta).$$

We follow section 2 of [AMR], using their terminology. We shall prove the following theorem:

Theorem 3.1 *Let D be a directed graph. Let k be the larger of the maximum indegree and the maximum outdegree (i.e. $k = \max(\Delta_{in}(D), \Delta_{out}(D))$), then*

$$dst(D) \leq k + 20 \log k + 84.$$

We let D be a directed graph, $k = \max(\Delta_{in}(D), \Delta_{out}(D))$, and $c = \lceil 5 \log k + 20 \rceil$. We find $k + c$ directed star forests such that the edges not covered by the forests form a subgraph H of D with $\mathrm{dst}(H) \leq 3c$. We shall need the following two lemmas, corresponding to lemmas 2.3 and 2.3 respectively.

Lemma 3.2 *If G is a directed graph (with possible multiple edges) such that every vertex has indegree at most c, then $dst(G) \leq 3c$.*

Proof: We can partition the edges of G into c graphs where the indegree of each vertex is at most one. We shall cover each of these by three directed stars. A directed graph of indegree at most one has connected components that have at most one cycle. We remove an edge from each cycle (this is our first star forest) and are left with a directed forest of indegree zero or one. This is covered by two directed star forests (taking every other vertex as the centers). ∎

We note that this is best possible by the following graph. Consider a directed cycle of length three. Replace each edge by c edges and we are left with a directed multigraph of indegree c. Each star forest can contain at most one edge, hence its directed star arboricity is $3c$.

Lemma 3.3 *Let D be a directed graph in which each vertex has indegree at most k. Suppose that for each vertex x we have chosen a subset L_x of $X = \{1, \ldots, k + c\}$ with $|L_x| \leq c$ such that the following property holds: for each vertex x, the family $(L_y : \overrightarrow{yx} \in E(D))$ has a transversal. Then we can partition the edges of D into $k + 4c$ directed star forests.*

(A *transversal* of a family $(A_i : i \in I)$ of sets is a family of distinct elements $(t_i : i \in I)$ with $t_i \in A_i$.)

Proof: There is a 'coloring' $f : E(D) \to X$ such that $f(\overrightarrow{yx}) \in L_y$, and for each vertex x the edges entering x have distinct colors. For each $i \in X$, let S_i be the set of edges $\overrightarrow{yx}$ colored i and such that $i \notin L_x$ so no edge in S_i can leave x. Thus S_i is a directed star forest. The S_i correspond to $k + c$ star forests. The edges not contained in the S_i are those $\overrightarrow{yx}$ such that

$f(\overrightarrow{yx}) \in L_x$. But at most c edges can enter any vertex x, so by Lemma 3.2 its directed star arboricity is at most $3c$ and the lemma follows. ∎

Proof of Theorem 3.1: Let D be a simple directed graph, and let $k = \max(\Delta_{in}(G), \Delta_{out}(G))$. Let $c = \lceil 5 \log k + 20 \rceil$. We must show that $\mathrm{dst}(D) \leq k + 4c$. If $k \leq c$, this is true by Lemma 3.2, as $\mathrm{dst}(D) \leq 3k$, so we assume that $k \geq c$.

For each vertex x of G, we assign a subset L_x of $X = \{1, \ldots, k + c\}$ by c independent random samplings of X. We let A_x be the event that the family $(L_y : \overrightarrow{yx} \in E(D))$ fails to have a transversal. In Lemma 2.5 [AMR], it is shown that $P(A_x) \leq k^{3-\frac{c}{2}}$. The event A_x is independent of all A_y for which there is no vertex z such that $\overrightarrow{zx}$ and $\overrightarrow{zy} \in E(D)$. The dependency graph for the events has degree at most k^2 and we may apply the Lovász Local Lemma ([EL], quoted as Lemma 2.4 in [AMR]). We restate the lemma here.

Lemma 3.4 (Lovász Local Lemma) *Let G be a simple graph on the vertex set $V = \{1, 2, \ldots, n\}$ with maximum degree $\Delta(G) \leq d$. A probability event A_i is associated with each vertex $i \in V$ such that $P(A_i) \leq 1/4d$ and the event A_i is independent of all A_j's for which j is not adjacent to i in G. Then $P(\bar{A}_1 \bar{A}_2 \cdots \bar{A}_n) > 0$.*

The hypotheses of this lemma are met, so there exists a family of sets satisfying the conditions of Lemma 3.3, and by this lemma the theorem is proved. ∎

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References

[AK] J. Akiyama and M. Kano, Path factors of a graph, in: *Graph theory and its Applications,* Wiley and Sons, New York, 1984.

[AA] I. Algor, N. Alon, The star arboricity of graphs, *Disc. Math.* **75** (1989) 11–22.

[AMR] N. Alon, C. McDiarmid, and B. Reed, Star arboricity, *Combinatorica* **12** (4) (1992) 375–380.

[BM] R. A. Brualdi and J. J. Q. Massey, Incidence and strong edge colorings of graphs, *Disc. Math.* **122** (1993) 51–58.

[EL] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in: *Infinite and Finite Sets*, A. Hajnal et al. editors, North Holland, Amsterdam, 1975.

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