

Most primitive groups are full automorphism groups of edge-transitive hypergraphs

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Abstract

We prove that, for a primitive permutation group G acting on a set X of size n , other than the alternating group, the probability that $\text{Aut}(X, Y^G) = G$ for a random subset Y of X , tends to 1 as $n \rightarrow \infty$. So the property of the title holds for all primitive groups except the alternating groups and finitely many others. This answers a question of M. H. Klin. Moreover, we give an upper bound $n^{1/2+\epsilon}$ for the minimum size of the edges in such a hypergraph. This is essentially best possible.

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primitive group, edge-transitive hypergraph

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Dedicated to the memory of Ákos Seress

1. Introduction

It is well known that, although every abstract group is the full automorphism group of a graph, not every permutation group is. Moreover, the alternating group is not the automorphism group of any family of sets, or

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even the intersection of automorphism groups of families of sets; for any set system admitting the alternating group admits the symmetric group. It is our purpose here to show that, at least for primitive groups, there are only finitely many other exceptions. Moreover, we can assume that G acts transitively on the sets of the system, and that the size of the sets is not too large.

Theorem 1.1. *For any $\epsilon > 0$, there is a finite list \mathcal{L} of primitive permutation groups such that the following holds for every k satisfying $n^{1/2+\epsilon} \leq k \leq n/2$. If G is a primitive permutation group of degree n , which is not the alternating group and not in the list \mathcal{L} , then there is a k -uniform hypergraph (X, \mathcal{B}) such that $\text{Aut}(X, \mathcal{B}) = G$ and G acts transitively on \mathcal{B} .*

Corollary 1.2. *Let G be a primitive group on X , not the alternating group and not one of a finite list of exceptions. Then there is a uniform hypergraph (X, \mathcal{B}) such that $\text{Aut}(X, \mathcal{B}) = G$ and G acts transitively on \mathcal{B} .*

If we do not restrict the edge size, then the G -orbits of almost all subsets of the vertex set define edge-transitive hypergraphs that have automorphism group no larger than G :

Theorem 1.3. *Let $|X| = n$, and let G be a primitive permutation group on X but not the alternating group. If Y is a random subset of X and Y^G the set of G -translates of Y then*

$$\text{Prob}(\text{Aut}(X, Y^G) > G) < \exp(-n^{1/2+o(1)}).$$

Remark 1. We give first the proof of Theorem 1.3, since Theorem 1.1 uses similar arguments but needs more refined estimates.

Remark 2. We have not attempted to determine the “finite list of exceptions” in Corollary 1.2. Note that any set-transitive group is an exception. There are just four of these apart from S_n and A_n , viz. the Frobenius group of order 20 ($n = 5$), $\text{PGL}(2, 5)$ ($n = 6$), $\text{PGL}(2, 8)$ and $\text{PGL}(2, 9)$ ($n = 9$). Another exception is the Frobenius group of order 21 ($n = 7$); any orbit (or union of orbits) of G on 3-sets (or on 4-sets) admits one of the three minimal overgroups of G (the Frobenius group of order 42 and one of two copies of $\text{PGL}(3, 2)$).

A very similar situation arose in connection with the main theorem of [9], where it was shown that every primitive group apart from symmetric and alternating groups and a finite list has a regular orbit on the power set of its domain. The finite list was computed by Seress [22]. His methods were our inspiration to complete the work reported here.

Remark 3. What is the least size of edges in a hypergraph (X, \mathcal{B}) with $\text{Aut}(X, \mathcal{B}) = G$? We cannot get by with edges of fixed size. Consider, for example, the alternating group G of degree m in its induced action on 2-sets. A k -subset Y of X is the edge set of a graph with m vertices and k edges. If \mathcal{B} is a collection of k -sets with $\text{Aut}(X, \mathcal{B}) = G$, then some set $Y \in \mathcal{B}$ does not admit any odd permutation in S_m ; so it has at most one fixed point, and at least $(m-1)/2 = \Omega(\sqrt{n})$ edges. This shows that Theorem 1.1 is best possible, apart from the value of ϵ in the exponent.

Our proof uses the following result [7]:

Lemma 1.4. *Let G be a primitive permutation group of degree n , other than S_n or A_n . Then either*

- (a) *G is S_m or A_m on 2-sets ($n = \binom{m}{2}$), or G is a subgroup of $S_m \text{ wr } S_2$ containing A_m^2 ($n = m^2$); or*
- (b) *$|G| \lesssim \exp(n^{1/3} \log n)$.*

We call G “large” or “small” according as the first or second alternative holds. Note that large groups have order roughly $\exp(n^{1/2} \log n)$. While Lemma 1.4 uses the classification of finite simple groups, a remarkable recent result by graduate students Xiaorui Sun and John Wilmes [23] (extending [1]) combined with [2] or [21] yields an elementary proof of a slightly weaker bound, namely, $\exp(n^{1/3}(\log n)^{7/3})$ in part (b), which would be just as adequate for our purposes.

Remark 4. This recent progress has not entirely eliminated our dependence on the classification of finite simple groups, and it is worth pointing out just how the classification is used. First of all, we actually require stronger bounds than Lemma 1.4 with a longer list of exceptions ([7], see Lemma 7.1). The best explicit result in this direction is due to Maróti [19], but we do not need the full force of this. We also use the classification of 2-homogeneous groups and the facts that simple groups can be generated by 2 elements and have small outer automorphism groups; but these could probably be avoided with care.

The results in this paper were mostly obtained during the Second Japan Conference on Graph Theory and Combinatorics at Hakone in 1990. We are also grateful to M. Deza, I. Faradžev, and M. H. Klin for asking persistent questions, and especially to Klin for proposing the question and for several contributions to the proof. The final steps in the argument (reported in Section 8) eluded us for some time, and so publication has been rather seriously delayed! In the intervening time, some of the results we used in the proof have been improved (for example in the papers [15, 19]), which allows a small amount of streamlining of our arguments; but we have kept the original arguments almost unchanged.

2. Outline of the proof

Throughout the paper we use the term “maximal subgroup” to mean “subgroup of S_n , maximal in the set of permutation groups other than S_n and A_n ”: that is, a maximal subgroup of S_n other than A_n , or a maximal subgroup of A_n contained in no other proper subgroup of S_n . We deduce the Theorem from the following result.

Lemma 2.1 (Main Lemma). *Let G be primitive on X with $|X| = n$, $G \neq S_n, A_n$. Then with probability $1 - \exp(-n^{1/2+o(1)})$, a random subset Y of X has the property that $M_Y = 1$ for every maximal subgroup M containing G .*

(Here M_Y denotes the setwise stabilizer of Y .)

The deduction of Theorem 1.3 from the Main Lemma runs as follows. Clearly the theorem holds for $G = S_n$; so we may assume $G \neq S_n, A_n$. Let Y be a subset for which the conclusion of the Main Lemma holds, and let $H = \text{Aut}(X, Y^G)$. Then $H \geq G$, and so $H \leq M$ for some maximal subgroup M containing G . Thus $M_Y = 1$, and so $H_Y = 1$, and

$$|G| \geq |Y^G| = |Y^H| = |H| \geq |G|,$$

from which $H = G$ follows.

Now to prove the Main Lemma, we need estimates for

- (a) the number of conjugacy classes of maximal subgroups;
- (b) for each conjugacy class, the number of subgroups in that class containing G ;
- (c) for each maximal subgroup M , the probability that $M_Y = 1$ for random Y .

The estimate for (a) is taken from [16], improving earlier bounds in [4], [20], and [15].

Lemma 2.2. *The number of conjugacy classes of maximal subgroups is at most $(\frac{1}{2} + o(1))n$.*

Estimates for (b) are given in the next two sections; we separate the cases of large and small maximal subgroups. In section 5 we prove a uniform bound $\exp(-c\sqrt{n})$ for (c), using elementary bounds on the minimal degree. In section 6 we do the accounting necessary to prove the Main Lemma. Finally in section 7, we indicate how to modify the argument in order to prove Theorem 1.1.

We will need the following result at two points in the proof.

Lemma 2.3. *A primitive group of degree n can be generated by at most $c(\log n)^2$ elements.*

PROOF. If G has abelian socle, then $|G| \leq n^{1+\log n}$ and the claim is clear. So let G have socle T^k , where T is non-abelian and simple. Then $G \leq T^k \cdot (\text{Out} T)^k \cdot S_k$. Now T^k requires at most $2k$ generators; a subgroup of $(\text{Out} T)^k$, at most $3k$ (since $\text{Out}(T)$ is at most 3-step cyclic); and a subgroup of S_k , at most $2k$ [3, 10]. Since $k \leq \log n$, the result holds.

3. Large maximal subgroups

Let $S_m^{(2)}$ denote the action of S_m on the $n = \binom{m}{2}$ pairs.

In this section, we estimate the number of subgroups $S_m^{(2)}$ or $S_m \text{ wr } S_2$ containing a given primitive group G . In the first case, we obtain a best possible result.

Lemma 3.1. *A primitive group of degree n is contained in at most one subgroup isomorphic to $S_m^{(2)}$.*

PROOF. Let S_m act on $\Delta = \{1, \dots, m\}$, and identify X with the set of 2-subsets of Δ . If $G \leq S_m$ on pairs and G is primitive on X , then certainly G is 2-homogeneous on Δ (transitive on unordered pairs). These groups are listed, for example, in [8, pp.194–197]. Below we give original sources.

- (a) Affine case: G has an elementary abelian normal subgroup N regular on Δ . Then N is intransitive on X ; so G cannot be primitive.

We note that this includes the case when G is not 2-transitive; indeed, in that case G has odd order and is therefore solvable; being primitive on Δ , its minimal normal subgroup is transitive and elementary abelian.

Therefore we are left with the cases of non-affine, doubly transitive groups G on Δ . The socle T of G is then a non-abelian simple group. If T is alternating of degree $k < m$ then $k \leq 7$ and $n \leq 15$ by Maillet's 19th century result [17]; we move these cases to item (iii) below. The case $T = A_m$ is trivial. In the remaining cases, T is either of Lie type (these cases were classified by Curtis, Kantor and Seitz [11]) or T is sporadic. We consider each case.

- (b) $G \cong \text{Sp}(2d, 2)$, $m = 2^{2d-1} \pm 2^{d-1}$. Then Δ is embedded in $\text{AG}(2d, 2)$ in a G -invariant fashion, and G preserves the restriction of the parallelism relation to 2-subsets of Δ ; so G cannot be primitive on X .
- (c) G preserves a Steiner system $S(2, k, m)$, $k > 2$. (This includes groups containing $\text{PSL}(d, q)$ with $d > 2$, unitary and Ree groups in Table 7.4, p.197, of [8].) Then the set of pairs contained in a block of the Steiner system is a block of imprimitivity for G on X ; once again G can not be primitive.
- (d) In the remaining cases, overgroups $S_m^{(2)}$ of G correspond bijectively to unions of orbital graphs of G which are isomorphic to the line graph of K_m . We show that, in each case, there is at most one such union.
 - (i) $Sz(q) \leq G \leq \text{Aut}(Sz(q))$, $m = q^2 + 1$. Let N be the Sylow 2-normalizer in $Sz(q)$. Then N fixes a point of Δ , say 1, and is transitive on the remaining points. Thus N has an orbit of length q^2 on X , consisting of pairs $\{1, i\}$ for $i \in \Delta \setminus \{1\}$. Any other N -orbit on X has length divisible by $q - 1$, by considering the 2-point stabilizer in $Sz(q)$. So the orbit of length q^2 is unique. Now this orbit must be a clique in the required graph, and the edge sets of its translates cover all edges of the graph. So the graph is unique.
 - (ii) $\text{PSL}(2, q) \leq G \leq \text{PTL}(2, q)$, $m = q + 1$. This case is similar to the preceding one.

- (iii) Finitely many others. These are handled by *ad hoc* methods. For example, the groups M_n , for $n = 11, 12, 23, 24$, are 4-transitive on Δ , and so have just two non-trivial orbital graphs, the line graph of K_m and its complement.

Lemma 3.2. *A primitive group of degree $n = m^2$ is contained in at most $n^{c \log n}$ subgroups isomorphic to $S_m \text{ wr } S_2$.*

PROOF. If G has an overgroup $M \cong S_m \text{ wr } S_2$, then G has a subgroup H of index 2 which has a block B of imprimitivity of size \sqrt{n} . The pair (H, B) determines M . (Indeed, it is easy to show that any such pair gives rise to an overgroup of the correct form.) So we have to estimate the number of such pairs.

- (a) G has at most $n^{c \log n}$ subgroups of index 2. This is immediate from Lemma 2.3.
- (b) A block of imprimitivity containing a point x for a transitive group H is determined by its stabilizer, a subgroup K of H containing H_x . Now K is generated by at most $\log n$ cosets of H_x , so there are at most $n^{\log n}$ choices for it.

Multiplying (a) and (b) yields the Lemma.

4. Small maximal subgroups

In this section we prove a general bound. It is good enough for our purposes for small maximal subgroups, but not for large ones.

Lemma 4.1. *Let G be primitive, M maximal, of degree n . Then G lies in at most $(c|M|)^{c \log^2 n}$ conjugates of M .*

PROOF. First, M contains at most $|M|^{c \log^2 n}$ copies of G , because G has at most $c \log^2 n$ generators, as we observed in Section 2.

Next, $|N_{S_n}(G) : G| \leq \exp(c \log^2 n)$. For $|N_{S_n}(G) : G|$ does not exceed $|\text{Out}(N)|$, where N is the socle of G . The bound is clear if N is abelian. Otherwise, $N = T^k$, where $k \leq \log n$; and $|\text{Out} N| = |\text{Out} T|^k k!$. If T is alternating, then $|\text{Out} T|$ is bounded by a constant; otherwise, $|\text{Out} T| \leq \log |T|$, and $|T| \leq n^{\log n}$. In either case, the bound holds.

Now let x be the number of conjugates of M containing G . Counting pairs (G', M') , where G' and M' are conjugates of G and M respectively, we obtain

$$x|S_n : N_{S_n}(G)| \leq |M|^{c \log^2 n} |S_n : M|.$$

Rearranging, $x|M : G| \leq |M|^{c \log^2 n} |N_{S_n}(G) : G|$, which gives the result (since $|M : G| \geq 1$).

Remark 5. This bound is probably much too large.

5. The probability of rigidity

The analysis in this section has been performed several times by different people for various applications. The *minimal degree* of G is the least number of points moved by a non-identity element of G . Doubly transitive groups other than S_n and A_n have minimal degree at least $n/4$ by Alfred Bochert's 1892 combinatorial gem [6]. Primitive but not doubly transitive groups have minimal degree at least $(\sqrt{n}-1)/2$ by elementary arguments [1, Thm 0.3], so the same lower bound¹ holds for all primitive groups other than S_n and A_n . For convenience we cite the following slightly stronger bound taken from [14].

Lemma 5.1. *A primitive permutation group of degree n , other than S_n or A_n , has minimal degree at least $(\sqrt{n})/2$.*

Lemma 5.2. *Let M be a maximal primitive group of degree n , acting on a set X . If Y is a random subset of X , then*

$$\text{Prob}(M_Y \neq 1) \leq \exp(-c\sqrt{n}),$$

for some constant c .

PROOF. Again we treat large and small groups separately. If M is large, we require the probability that a random graph (or bipartite graph) admits a non-trivial automorphism, for which estimates exist [13].

So suppose M is small. Let m be its minimal degree. If $g \in M$, $g \neq 1$ then g has at most $n - m/2$ cycles on X (the extreme case occurring if g is

¹Still by elementary arguments, the lower bound $2\sqrt{n}$ holds for the minimal degree of all primitive groups other than S_n and A_n , for all sufficiently large n [23].

an involution moving m points), and so g fixes at most $2^{n-m/2}$ subsets. So the probability that a random subset is fixed by g is at most $2^{-m/2} \leq 2^{-\sqrt{n}/4}$. Then

$$\begin{aligned} \text{Prob}(M_Y \neq 1) &\leq |M| \cdot 2^{-\sqrt{n}/4} \\ &\leq \exp(n^{1/3} \log n) \cdot 2^{-\sqrt{n}/4} \\ &\leq \exp(-c\sqrt{n}). \end{aligned}$$

6. Completion of the Proof

Now by the above Lemmas, the number $F(G)$ of maximal subgroups containing G is at most

$$1 + n^{c \log n} + \exp(\log^4 n) \cdot (c \exp(n^{1/3} \log n))^{c \log^2 n},$$

and the probability that the conclusion of the Main Lemma fails is at most $F(G) \exp(-c\sqrt{n})$. So the result is proved.

7. Bounding the size of sets required

The proof of Theorem 1.1 follows closely the argument we have given, but the technical details are considerably harder. The difficulty arises because the analogue of Lemma 5.2 is much weaker. If M is maximal primitive with minimal degree m , then a non-identity element of M has at most $n - m/2$ cycles, and so fixes at most

$$\sum_{i=0}^k \binom{n - m/2}{i} \leq 2 \binom{n - m/2}{k}$$

subsets of size k . So

$$\begin{aligned} \text{Prob}(M_Y \neq 1) &\leq 2|M| \binom{n - m/2}{k} / \binom{n}{k} \\ &\leq 2|M| \left(1 - \frac{m}{2n}\right)^k \\ &< 2|M| \exp\left(-\frac{km}{2n}\right). \end{aligned}$$

Now $|M| \geq 2^{n/m}$ (this bound holds for any transitive group of degree n and minimal degree m [14, 5]), and so

$$\begin{aligned} \text{Prob}(M_Y \neq 1) &\leq |M| \exp\left(-\frac{k}{2 \log |M|}\right) \\ &= 2 \exp\left(\log |M| - \frac{ck}{\log |M|}\right). \end{aligned}$$

For $k = n^{1/2+\epsilon}$, no conclusion is possible unless $|M| \leq \exp(n^{1/4-\epsilon})$.

Fortunately the classification of finite simple groups gives such a bound with known exceptions. For this result see [7, 19], but note that we do not need the full refinement of Maróti's estimates.

Lemma 7.1. *If G is primitive of degree n and maximal, then either*

- (a) *G is contained in S_m on k -sets, $k = 2, 3$ or 4 , ($n = \binom{m}{k}$), or $S_m \text{ wr } S_k$, $k = 2, 3$, or 4 ($n = m^k$); or*
- (b) $|G| \leq \exp(n^{1/5} \log n)$.

The groups contained in $S_m \text{ wr } S_2$, where S_m acts on 2-sets (with order around $\exp(n^{1/4} \log n)$) do not need to be considered since they are contained in $S_{m(m-1)/2} \text{ wr } S_2$.

If we redefine “large” maximal subgroups to include all those under (a), then our estimates suffice for “small” maximal subgroups, and we can use separate estimates for the probabilities that random k -uniform hypergraphs and random k -partite k -uniform hypergraphs, for $k = 2, 3, 4$, admit non-trivial automorphisms. These are given in the next Section. We also need to prove analogues of Lemmas 3.1 and 3.2 for $k = 3, 4$.

Lemma 7.2. *A primitive group of degree n lies in at most one copy of S_m on k -sets, for fixed $k \geq 2$ and $m > 2k$.*

PROOF. The case $k = 2$ is Lemma 3.1. For $k \geq 3$, there are very few groups other than S_m and A_m which act primitively on k -sets. For such a group must be k -homogeneous, and hence 2-transitive [18]; as before, it cannot have a regular normal subgroup N (since N would be intransitive on k -sets). This leaves only the cases $\text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$ (with $k = 3$) or G is a Mathieu group. In the first case, the stabiliser of a 3-set is not maximal (by inspection of the list of maximal subgroups in [12]). The Mathieu groups are handled by *ad hoc* methods.

Lemma 7.3. *A primitive group of degree n lies in at most $n^{c \log n}$ copies of $S_m \text{ wr } S_k$, for fixed $k \geq 2$ and $m > 2$.*

PROOF. The proof of Lemma 3.2 applies with trivial changes.

8. Asymmetry of random hypergraphs

Lemma 8.1. *For $2 \leq t \leq n/2$, let P_t denote the probability that a random t -uniform hypergraph on n vertices is not asymmetric. For $t = 2$ we have $P_2 = \sqrt{2}n^2 2^{-n/2}(1 + o(1))$. For $t \geq 3$ we have $P_t < \exp(-c_1 \binom{n-1}{t-1}) < 2^{-cn^2}$ for some positive absolute constants c, c_1 .*

PROOF. (For $t = 2$ this is well known [13], but we include this case, too, in the proof.) If the random hypergraph \mathcal{H} has a nonidentity automorphism then it has one of prime order. Let σ be a permutation of $V(\mathcal{H})$ of prime order p . Let $N(\sigma)$ denote the number of t -sets moved by σ , so that $N(\sigma) \leq \binom{n}{t}$. The number of cycles of σ on t -sets is at most $\binom{n}{t} - N(\sigma)/2$, the extremal case being where σ is an involution.

Let $P(\sigma)$ denote the probability that σ is an automorphism of \mathcal{H} . We think of \mathcal{H} being generated by flipping a coin for every t -tuple $F \subset V$ to decide whether or not to include F in $E(\mathcal{H})$. However, for each σ -orbit of t -sets, we can flip the coin only once. Therefore $P(\sigma) \leq 2^{-N(\sigma)/2}$ and

$$P_t \leq \sum_{\sigma} 2^{-N(\sigma)/2}, \quad (1)$$

where the summation extends over all permutations σ of prime order.

Let s denote the size of the support of σ (number of elements of $V(\mathcal{H})$ moved). Note that $p|s$. Let $\rho(s, \ell, p) = \binom{s/p}{\ell/p}$ if p divides ℓ , and 0 otherwise. Let us compute $N(\sigma)$ by counting for each $\ell \geq 1$ those t -sets which intersect the support of σ in exactly ℓ elements. Adding these up we obtain

$$N(\sigma) = \sum_{\ell=1}^t \left(\binom{s}{\ell} - \rho(s, \ell, p) \right) \binom{n-s}{t-\ell}.$$

This quantity is estimated as

$$N(\sigma) \geq \frac{1}{2} \sum_{\ell=1}^t \binom{s}{\ell} \binom{n-s}{t-\ell} = \frac{1}{2} \left(\binom{n}{t} - \binom{n-s}{t} \right). \quad (2)$$

For $t = 2$, the right hand side is

$$N(\sigma) \geq \frac{1}{2} \left(\binom{n}{2} - \binom{n-s}{2} \right) = s(n-s/2-1/2)/2.$$

From equation (1) we then infer that

$$P_2 \leq \sum_{s=2}^n \binom{n}{s} (s! - 1) 2^{-s(n-s/2-1/2)/4} = \binom{n}{2} 2^{(-n+3/2)/2} (1 + o(1)),$$

as stated. The opposite inequality follows by taking the second term of the inclusion-exclusion formula into account in calculating the probability that there exist two vertices switched by a transposition.

For $t \geq 3$, equation (2) implies $N(\sigma) > \binom{n-1}{t-1} > c_2 n^2$, therefore

$$P_t \leq n! 2^{-(1/2) \binom{n-1}{t-1}}.$$

A t -partite *transversal hypergraph* \mathcal{H} is a t -uniform hypergraph whose vertex set is partitioned into t “layers” $V(\mathcal{H}) = V_1 \cup \dots \cup V_t$ and each edge intersects each class V_i in exactly one element. The partition into layers is given and is part of the definition of \mathcal{H} . Automorphisms of \mathcal{H} preserve the partition by definition (but may interchange the layers). A transversal hypergraph is *balanced* if all layers have equal size.

Lemma 8.2. *For $2 \leq t \leq n/2$, let Q_t denote the probability that a random balanced t -partite transversal hypergraph on $n = tr$ vertices is not asymmetric. For $t = 2$ we have $Q_2 = cn^2 2^{-n/4} (1 + o(1))$. For $t \geq 3$ we have $Q_t < \exp(-c_1 \binom{n-1}{t-1}) < 2^{-cn^2}$ for some positive absolute constants c, c_1 .*

PROOF. As before, let σ be a permutation of $V = V(\mathcal{H})$ of prime order p . By our remark about automorphisms before the lemma, σ respects the partition into layers (V_1, \dots, V_t) . We say that a t -tuple is *transversal* if it intersects each layer in exactly one element. The total number of transversal t -sets is r^t where $r = n/t$. Let $N(\sigma)$ denote the number of transversal t -sets moved by σ . Let $Q(\sigma)$ denote the probability that σ is an automorphism of the random transversal hypergraph \mathcal{H} . Just as in equation (1), we have

$$Q_t \leq \sum_{\sigma} 2^{-N(\sigma)/2}, \quad (3)$$

where the summation extends over all permutations σ of prime order, respecting the partition (V_1, \dots, V_t) .

If σ moves some of the layers then it moves at least p layers. Hence in this case,

$$N(\sigma) \geq r^t - r^{t-p+1} > r^t/2. \quad (4)$$

If σ fixes all layers, then let s_i denote the number of elements in V_i that are moved by σ . So the support of σ has size $s = s_1 + \dots + s_t$.

The transversal t -sets not moved by σ are now exactly those which do not intersect the support of σ . Therefore

$$N(\sigma) = r^t - \prod_{i=1}^t (r - s_i) \geq r^t (1 - e^{-s/r}) \geq r^t \cdot \frac{s}{r} \cdot \left(1 - \frac{s}{2r}\right). \quad (5)$$

We infer that for $s \leq r/2$ we have

$$N(\sigma) \geq r^t 3s/(4r) \geq r^{t-1}, \quad (6)$$

and for $s \geq r/2$ we have (for $r \geq 3$)

$$N(\sigma) \geq r^t (1 - e^{-1/2}) > 0.39 \cdot r^t > r^{t-1}. \quad (7)$$

For $t = 2$ we conclude, separating the case $s = 2$, that

$$Q_2 \leq \left(\left(\frac{n}{2}\right)!\right)^2 2^{-n^2/8} + \frac{n(n-2)}{4} 2^{-n/4} + \sum_{s=4}^{n/4} \binom{n}{s} 2^{-3ns/16} + \sum_{s=n/4}^n \binom{n}{s} 2^{-0.39n^2/2}.$$

It is clear that the second term dominates this sum.

For $t \geq 3$ we obtain $Q_t \leq n! 2^{-r^{t-1}/2}$.

9. Open problems

1. Can we eliminate the need for the classification of finite simple groups from the proof of Theorem 1.1?

2. Assuming G is a primitive group whose socle is not a product of alternating groups, is it possible to reduce the size of the hyperedges in Theorem 1.1 below $n^{0.49}$ (for sufficiently large n)? Is it possible to reduce it to $n^{o(1)}$?

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